Communications in Statistics—Theory and Methods, 34: 475–484, 2005

ISSN: 0361-0926 print/1532-415X online
DOI: 10.1081/STA-200047430

Taylor & Francis
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Survival Analysis and Reliability

A Note on the Mean Residual Life Function of a Parallel System

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One of the most important types of system structures is the parallel structure. In the present article, we propose a definition for the mean residual life function of a parallel system and obtain some of its properties. The proposed definition measures the mean residual life function of a parallel system consisting of n identical and independent components under the condition that $n-i$, $i=0,2,\ldots,n-1$, *components of the system are working and other components of the system have already failed. It is shown that, for the case where the components of the system have increasing hazard rate, the mean residual life function of the system is a nonincreasing function of time. Finally, we will obtain an upper bound for the proposed mean residual life function.*

Keywords Exponential distribution; Mean residual life function; Order statistics; Parallel systems.

Mathematics Subject Classification Primary 62E10; Secondary 60E05.

1. Introduction

An important method of improving the reliability of a system is to build redundancy to it. A common structure of redundancy is the k-out-of-n systems and an important special case of k-out-of-n systems is parallel systems. In the last few years, numerous papers have appeared in statistical and reliability journals investigating the redundancy importance of components in complex systems by using the parallel system, series system, or, in general a k-out-of-n system (see e.g., Boland et al., 1991; Xie and Lai, 1996; Kuo and Prasad, 2000). A parallel system $\mathscr{S}_n$, consisting of n components, is a system which functions if and only if at least one of its n components functions. Let $T_1,\ldots,T_n$ denote the lifetimes of n components

Received July 18, 2003, Accepted July 7, 2004
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connected in a system with parallel structure. We assume that T_i's are continuous, independent, and identically distributed random variables with common distribution function F and survival function (or reliability function) $\overline{F} = 1 - F$. Let also $T_{1:n} \leq T_{2:n} \leq \cdots \leq T_{n:n}$ be the ordered lifetimes of the components. Then $T_{n:n}$ represents the lifetime of parallel system $\mathcal{S}_n$ with n components. If we denote the survival function of the system at time t by $\overline{S}(t)$, we have

$$\begin{aligned}\overline{S}(t) &= P(T_{n:n} \geq t)\\ &= 1 - F^n(t), \quad t > 0.\end{aligned} \tag{1.1}$$

Assuming that each component of the system has survived up to time t, the survival function of $T_i - t$ given that $T_i > t$, $i = 1, \ldots, n$, is

$$\overline{F}(x|t) = \frac{\overline{F}(t+x)}{\overline{F}(t)}. \tag{1.2}$$

This is the corresponding conditional survival function of the components at age t. From 1.2 we get that the mean residual life (MRL) function M of each component is equal to

$$M(t) = E(T_i - t|T_i \geq t) = \int_0^\infty \overline{F}(x|t)dx = \frac{\int_t^\infty \overline{F}(x)dx}{\overline{F}(t)}, \qquad i = 1, 2, \ldots, n.$$

The MRL function plays an important role in reliability and survival analysis. It is well known that the MRL function m characterizes the distribution function F uniquely (see, for example, Kotz and Shanbhag, 1980).

The MRL function of a system is closely related to the MRL of its components. In the literature the MRL function of a parallel system is defined as

$$E(T_{n:n} - t|T_{n:n} > t)$$

and some properties of that have been obtained by, among others, Abouammoh and El-Neweihi (1986).

Recently, Bairamov et al. (2002), under the condition that none of the components of the system fails at time t, defined the MRL of the parallel system $\mathcal{S}_n$ as

$$M^1_{(n)}(t) = E(T_{n:n} - t|T_{1:n} > t) \tag{1.3}$$

and obtained several properties of $M^1_{(n)}(t)$. They have also shown that, under some regularity conditions, the survival function $\overline{F}$ can be represented as

$$\overline{F}(t) = \exp\left\{-\frac{1}{n}\int_0^t \frac{1 + \frac{d}{dx}M^1_{(n)}(x)}{M^1_{(n)}(x) - M^1_{(n-1)}(x)}dx\right\}, \tag{1.4}$$

where $M^1_{(n-1)}(t)$ is the MRL of the parallel system $\mathcal{S}_{n-1}$ having $n-1$ components.

The aim of the present article is to extend the definition of the MRL function proposed by Bairamov et al. (2002) and explore its properties. We will define the MRL function of a system, under the condition that $T_{r:n} > t$, i.e., $(n - r + 1)$,

$r = 1, 2, \ldots, n$, components of the system are still working. In other words, we define the MRL of the parallel system $\mathcal{S}_n$ as

$$M^r_{(n)}(t) = E(T_{n:n} - t | T_{r:n} > t), \quad r = 1, 2, \ldots, n. \tag{1.5}$$

We will show that $M^r_{(n)}(t)$, for fixed n, is a decreasing function of r, $r = 1, \ldots, n$. We will also show that, for the case where $r = 1$, $M^r_{(n)}(t)$ is an increasing function of n. If the components of the system have an increasing hazard rate, it is also shown that $M^r_{(n)}(t)$ is a nonincreasing function of t. Finally, we will obtain an upper bound for $M^r_n(t)$.

2. Main Results

Let $M^r_{(n)}(t)$ be the MRL function of the parallel system $\mathcal{S}_n$. We assume that the lifetime of the components of the system are independent and identically distributed with common distribution F. In the following theorem, we first obtain a representation formula for $M^r_{(n)}(t)$.

Theorem 2.1. *If $M^r_{(n)}(t)$ is the MRL of the parallel system $\mathcal{S}_n$ defined as* (1.5), *then for* $\overline{F}(t) > 0$

$$M^r_{(n)}(t) = \frac{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t) \sum_{j=1}^{n-i} (-1)^{j+1} \binom{n-i}{j} M_j(t)}{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t)} \quad r = 1, 2, \ldots, n \ t > 0,$$

where $M_j(t) = \frac{\int_t^\infty \overline{F}^j(x)dx}{\overline{F}^j(t)}$, $\phi(t) = \frac{F(t)}{\overline{F}(t)}$.

Proof. To prove the theorem first note that we have

$$\begin{aligned} &P(T_{n:n} > x + t, T_{r:n} > t) \\ &= \sum_{i=0}^{r-1} \binom{n}{i} F^i(t) \sum_{j=1}^{n-i} (-1)^{j+1} \binom{n-i}{j} \overline{F}^{n-i-j}(t) \overline{F}^j(x+t). \end{aligned} \tag{2.1}$$

Hence, if $S(t|x)$ denotes the conditional survival of $T_{n:n}$ at $x + t$ given that $T_{r:n}$ is greater than t, then

$$\begin{aligned} S(x|t) &= P(T_{n:n} > x + t | T_{r:n} > t) \\ &= \frac{P(T_{n:n} > x + t, T_{r:n} > t)}{P(T_{r:n} > t)} \\ &= \frac{\sum_{i=0}^{r-1} \binom{n}{i} F^i(t) \sum_{j=1}^{n-i} (-1)^{j+1} \binom{n-i}{j} \overline{F}^{n-i-j}(t) \overline{F}^j(x+t)}{\sum_{i=0}^{r-1} \binom{n}{i} F^i(t) \overline{F}^{n-i}(t)}. \end{aligned} \tag{2.2}$$

Therefore,

$$\begin{aligned} M^r_{(n)}(t) &= \int_0^\infty S(x|t)dx \\ &= \frac{\sum_{i=0}^{r-1} \binom{n}{i} F^i(t) \sum_{j=1}^{n-i} (-1)^{j+1} \binom{n-i}{j} \overline{F}^{n-i-j}(t) \int_0^\infty \overline{F}^j(t+x)dx}{\sum_{i=0}^{r-1} \binom{n}{i} F^i(t) \overline{F}^{n-i}(t)}. \end{aligned} \tag{2.3}$$

Hence, on taking $M_j(t) = \frac{\int_t^\infty \overline{F}^j(x)dx}{\overline{F}^j(t)}$ and $\phi(t) = \frac{F(t)}{\overline{F}(t)}$ we get

$$M_{(n)}^r(t) = \frac{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t) \sum_{j=1}^{n-i} (-1)^{j+1} \binom{n-i}{j} M_j(t)}{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t)}. \tag{2.4}$$

Remark 2.1. To define the MRL $M_{(n)}^r(t)$ and obtain (2.4), one does not actually need to restrict the support of F on $(0, \infty)$. In general, $M_{(n)}^r(t)$ can be defined for the distribution functions with left extremity $-\infty \leq a$ and right extremity $b \leq \infty$, respectively.

Remark 2.2. In Theorem 2.1, if we take $r = 1$ then

$$M_{(n)}^1(t) = E(T_{n:n} - t | T_{1:n} > t) = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} M_i(t). \tag{2.5}$$

Hence, $M_{(n)}^r(t)$ can be written as

$$M_{(n)}^r(t) = \frac{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t) M_{(n-i)}^1(t)}{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t)}. \tag{2.6}$$

That is, $M_{(n)}^r(t)$ can be represented as a convex combination of $M_{(n-i)}^1(t)$, $i = 0, 1, \ldots, r-1$.

Remark 2.3. Let $A_1, \ldots, A_n$ be n events. Then the representation formula for $M_{(n)}^1(t)$ in (2.5) is an analog of the well-known Boole formula, based on the principle of inclusion and exclusion, for the probability of occurrence of at least one of the A_i, $i = 1, \ldots n$. See David and Nagaraja (2003, p. 125).

Remark 2.4. Let $g_n(t) = E\{T_{n:n} \mid T_{1:n} = t\}$ be the regression of $T_{n:n}$ on $T_{1:n}$ which can be interpreted as the best predictor of a parallel system consisting of n identical and independent components knowing the time of the first failure, i.e., the time when the weakest component fails. It is not difficult to observe that

$$g_n(t) = \frac{n-1}{\overline{F}^{n-1}(t)} \int_t^\infty y(F(y) - F(t))^{n-2} f(y)dy = M_{(n-1)}^1 + t. \tag{2.7}$$

Therefore, $M_{(n)}^r(t)$ can be also represented as

$$M_{(n)}^r(t) = \frac{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t) \left[g_{n+1-i}(t) - t\right]}{\sum_{i=0}^{r-1} \binom{n}{i} \phi^i(t)}. \tag{2.8}$$

Regression of order statistics aroused interest of many statistician in recent years. A characterization of distributions via linearity of regression of order statistics was considered first by Ferguson (1967). Dembinska and Wesolowski (1998) give a complete solution for non adjacent order statistics.

In the following theorem, we will show that $M^1_{(n)}(t)$ is an increasing function of n for any $t > 0$.

Theorem 2.2. $M^1_{(n)}(t) \geq M^1_{(n-1)}(t)$, *for any* $t > 0$.

Proof. We have

$$M^1_{(n)}(t) - M^1_{(n-1)}(t) = \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}M_i(t) - \sum_{i=1}^{n-1}(-1)^{i+1}\binom{n-1}{i}M_i(t)$$
$$= \sum_{i=0}^{n-1}(-1)^{i}\binom{n-1}{i}M_{i+1}(t). \tag{2.9}$$

Hence, to prove the result, we need to show that the right-hand side of Eq. (2.9) is greater than or equal to zero. This is equivalent to show that

$$\sum_{i=0}^{n-1}(-1)^{i}\binom{n-1}{i}\overline{F}^{n-i-1}(t)\int_t^\infty \overline{F}^{i+1}(x)dx \geq 0. \tag{2.10}$$

Note that, for any $t > 0$, the left-hand side of (2.10) can be written as

$$\int_t^\infty \left(\overline{F}^{n-1}(t) - \binom{n-1}{1}\overline{F}^{n-2}(t)\overline{F}(x) + \binom{n-1}{2}\overline{F}^{n-3}(t)\overline{F}^2(x)\right.$$
$$\left. - \cdots (-1)^{n-1}\overline{F}^{n-1}(x)\right)\overline{F}(x)dx \tag{2.11}$$

which is equal to

$$\int_t^\infty \left(\overline{F}(t) - \overline{F}(x)\right)^{n-1}\overline{F}(x)dx \qquad t > 0. \tag{2.12}$$

Since $\overline{F}$ is a non increasing function, the integrant is non negative. This implies that (2.12) is non negative. Hence, we get

$$M^1_{(n)}(t) \geq M^1_{(n-1)}(t).$$

The theorem simply says that the MRL of a parallel system having n components, under the condition that all the components of the system are working, is greater than or equal to the MRL of a parallel system having $n-1$ components, under the same condition.

Intuitively, one expects that a parallel system consisting of n components, in which at time t, $(n-r+1)$ of its components are still working have, in average, more lifetime than a parallel system consisting of n components of which $(n-r)$ components are still working. The following theorem proves this. In fact, the theorem shows that, for any $t > 0$, $M^r_{(n)}(t)$ is a decreasing function of r.

Theorem 2.3. $M^{r-1}_{(n)}(t) \geq M^r_{(n)}(t)$, *for* $r = 1, \ldots, n$, *and* $t > 0$.

Proof. We have

$$
\begin{aligned}
M_{(n)}^{r-1}(t) - M_{(n)}^{r}(t) &= \frac{\sum_{i=0}^{r-2}\binom{n}{i}\phi^i(t)M_{(n-i)}^1(t)}{\sum_{i=0}^{r-2}\binom{n}{i}\phi^i(t)} - \frac{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)M_{(n-i)}^1(t)}{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)} \\
&= \Big(\sum_{i=0}^{r-2}\binom{n}{i}\phi^i(t)\big[-\binom{n}{r-1}\phi^{r-1}(t)M_{(n-r+1)}^1\big] \\
&\quad + \binom{n}{r-1}\phi^{r-1}(t)\sum_{i=0}^{r-2}\binom{n}{i}\phi^i(t)M_{(n-i)}^1(t)\Big)\Big/ \\
&\quad \Big(\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t))(\sum_{i=0}^{r-2}\binom{n}{i}\phi^i(t)\Big) \\
&= \frac{\binom{n}{r-1}\phi^{r-1}(t)\big(\sum_{i=0}^{r-2}\binom{n}{i}\phi^i(t)(M_{(n-i)}^1 - M_{(n-r+1)}^1(t))\big)}{\Big(\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)\Big)\Big(\sum_{i=0}^{r-2}\binom{n}{i}\phi^i(t)\Big)}. \qquad (2.13)
\end{aligned}
$$

Note that for $i = 1, \ldots r-1$, we have $n - i \geq n - r + 1$. It is shown, in Theorem 2.2, that $M_{(n)}^1 \geq M_{(n-1)}^1$. This implies that $M_{(n-i)}^1 \geq M_{(n-r+1)}^1$, which in turn implies that the right-hand side of (2.13) is non negative. Hence we have the result.

In reliability theory, modeling, and the study of the properties of a lifetime random variable, several classes of distributions have been defined. One important class of life distributions is the class of increasing failure rate (IFR) distributions. An absolutely continuous distribution function F is said to belong to class of IFR distributions if the corresponding hazard rate $r(t) = \frac{f(t)}{\overline{F}(t)}$ is non decreasing in t. We refer the reader to Barlow and Proschan (1975) for more details on this. In the following theorem, we prove a result showing that when the components of the system have a common IFR distribution then $M_{(n)}^r(t)$ is decreasing in t.

Theorem 2.4. *Let $T_1, \ldots, T_n$ denote the lifetimes of the components of parallel system $\mathcal{S}_n$. Assume that T_i's are independent and have identical absolutely continuous distribution function F. If F is IFR, then $M_{(n)}^r(t)$ is a decreasing function of t.*

Proof. The assumption that F is IFR implies that for non negative values x and t, the ration $\frac{\overline{F}(x+t)}{\overline{F}(t)}$ is a decreasing function in t. This in turn implies that for $0 < t_1 < t_2$ and $n = 1, 2, \ldots,$

$$
\begin{aligned}
M_{(n)}^1(t_1) &= \int_0^\infty \left(1 - \left(1 - \frac{\overline{F}(x+t_1)}{\overline{F}(t_1)}\right)^n\right)dx \\
&\geq \int_0^\infty \left(1 - \left(1 - \frac{\overline{F}(x+t_2)}{\overline{F}(t_2)}\right)^n\right)dx \\
&= M_{(n)}^1(t_2). \qquad (2.14)
\end{aligned}
$$

That is, $M_{(n)}^1(t)$ is a decreasing function of t. Now, using this we show that

$$
M_{(n)}^r(t) = \frac{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)M_{(n-i)}^1(t)}{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)}. \qquad (2.15)
$$

is a decreasing function of t. Since $\phi(t)$ is an increasing function of t, without loss of generality we assume that $\phi(t) = t$. Hence we show that

$$K(t) = \frac{\sum_{i=0}^{r-1} \binom{n}{i} t^i M^1_{(n-i)}(t)}{\sum_{i=0}^{r-1} \binom{n}{i} t^i}$$

is a decreasing function of t. We have, on differentiating both sides of this last equation,

$$\begin{aligned} K'(t) = {} & \Big(\big(\textstyle\sum_{i=0}^{r-1} i\binom{n}{i} t^{i-1} M^1_{(n-i)}(t)\big)\big(\textstyle\sum_{i=0}^{r-1} \binom{n}{i} t^i\big) \\ & -\big(\textstyle\sum_{i=0}^{r-1} \binom{n}{i} t^i M^1_{(n-i)}(t)\big)\big(\textstyle\sum_{i=0}^{r-1} \binom{n}{i} i t^{i-1}\big)\Big)\Big/\big(\textstyle\sum_{i=0}^{r-1} \binom{n}{i} t^i\big)^2 \\ & + \frac{\big(\sum_{i=0}^{r-1} \binom{n}{i} t^i (M^1_{(n-i)}(t))'\big)\big(\sum_{i=0}^{r-1} \binom{n}{i} t^i\big)}{\big(\sum_{i=0}^{r-1} \binom{n}{i} t^i\big)^2}. \end{aligned} \tag{2.16}$$

As $M^1_{(n-i)}(t)$ is decreasing in t, the second term in the right-hand side is non positive. Hence, we need to show that the first term in the right-hand side is non positive. It can be shown after some manipulations that

$$\begin{aligned} & \left(\sum_{i=0}^{r-1} i\binom{n}{i} t^{i-1} M^1_{(n-i)}(t)\right)\left(\sum_{i=0}^{r-1} \binom{n}{i} t^i\right) - \left(\sum_{i=0}^{r-1} \binom{n}{i} t^i M^1_{(n-i)}(t)\right)\left(\sum_{i=0}^{r-1} i\binom{n}{i} t^{i-1}\right) \\ & = \sum_{j=0}^{r-1} \binom{n}{j} \sum_{i=j}^{r-1} (i-j) \binom{n}{i} t^{i+j-1} \big(M^1_{(n-i)}(t) - M^1_{(n-j)}(t)\big). \end{aligned} \tag{2.17}$$

In Theorem 2.2, it is shown that $M^1_{(n)}(t)$ is an increasing function of n for $n = 1, 2, \ldots$. This implies that $M^1_{(n-i)}(t) \le M^1_{(n-j)}(t)$ for $j \le i$ which in turn shows that the sum is non positive. Hence we get that $K'(t) \le 0$, i.e., $M^r_{(n)}(t)$ is a decreasing function of t. This completes the proof of the theorem.

Example 2.1. Let $T_1, \ldots, T_n$ denote the lifetimes of the components of parallel system $\mathcal{S}_n$. If $T_i's$ have exponential distribution with mean λ respectively, then using the above theorem, $M^r_{(n)}(t)$ is a non increasing function of t. In fact, it can be shown, in this case, that $M^r_{(n)}(t)$ has the following form:

$$\begin{aligned} M^r_{(n)}(t) &= \frac{\sum_{i=0}^{r-1} \binom{n}{i} (e^{\lambda t} - 1)^i \sum_{j=1}^{n-i} (-1)^{j+1} \binom{n-i}{j} \frac{1}{\lambda j}}{\sum_{i=0}^{r-1} \binom{n}{i} (e^{\lambda t - 1})^i} \\ &= \frac{\sum_{i=0}^{r-1} \binom{n}{i} (e^{\lambda t} - 1)^i \sum_{j=1}^{n-i} \frac{1}{\lambda j}}{\sum_{i=0}^{r-1} \binom{n}{i} (e^{\lambda t} - 1)^i}. \end{aligned} \tag{2.18}$$

In Fig. 1, we have presented the graph of the MRL function of a system containing five components in which the lifetime of the components are assumed to be exponential distribution with mean 1. It is seen that when r increases then the MRL of the system decreases and also $M^r_{(n)}(t)$ is a decreasing function of t.

Remark 2.5. Some other examples which are satisfied with the result of Theorem 2.4 are the Weibull and Gamma distributions with shape parameter greater than 1,

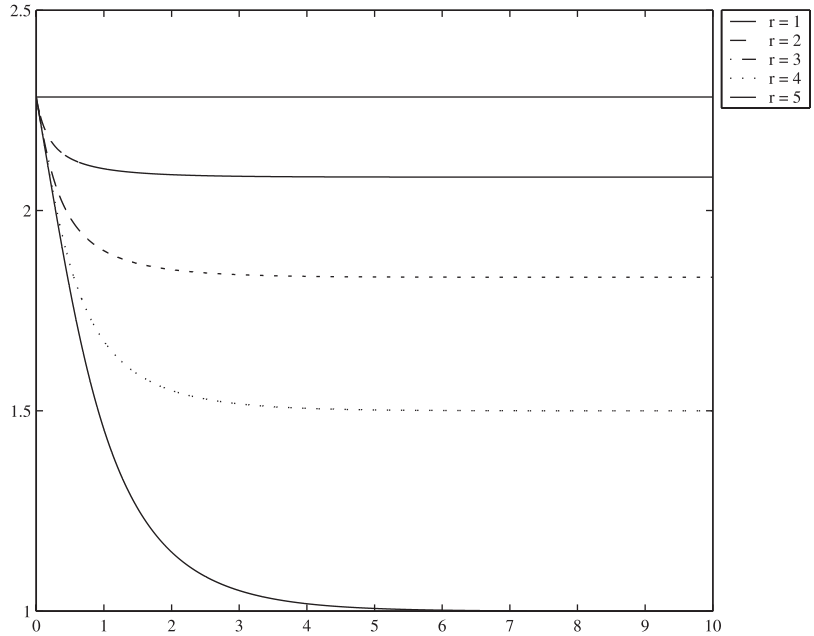

Figure 1. The MRL of a parallel system with $n = 5$ exponential components.

respectively. That is, if the components of the system are distributed as such distributions then the corresponding MRL functions $M^r_{(n)}(t)$ of the system are decreasing functions of time.

In the following theorem we find an upper bound for $M^r_{(n)}(t)$.

Theorem 2.5. *Let $M^r_{(n)}(t)$ denote the MRL of the parallel system $\mathcal{S}_n$. Then,*

$$M^r_{(n)}(t) \leq nM_1(t) - (n-1)M_2(t),$$

for any $t > 0$, and $r = 1, 2, \ldots, n$ where $M_1(t) = \frac{\int_t^\infty \overline{F}(x)dx}{\overline{F}(t)}$ and $M_2(t) = \frac{\int_t^\infty \overline{F}^2(x)dx}{\overline{F}^2(t)}$.

Proof. To prove the theorem, we first show that the result is valid for $r = 1$. To do this, we have to show that

$$M^1_{(n)}(t) = \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}M_i(t) \leq nM(t) - (n-1)M_2(t). \tag{2.19}$$

This is equivalent to show that

$$\sum_{i=3}^{n}(-1)^{i+1}\binom{n}{i}M_i(t) \leq \left(\binom{n}{2} - (n-1)\right)M_2(t),$$

or

$$\sum_{i=3}^{n}(-1)^i\binom{n}{i}M_i(t)+\frac{(n-1)(n-2)}{2}M_2(t)\geq 0.$$

Note that the last inequality is equivalent to

$$\frac{\int_t^\infty\left(\binom{n-1}{2}\overline{F}^{n-2}(t)-\binom{n}{3}\overline{F}(x)\overline{F}^{n-3}(t)+\cdots+(-1)^n\overline{F}^{n-2}(x)\right)\overline{F}^2(x)dx}{\overline{F}^n(t)}\geq 0. \tag{2.20}$$

But, it can be easily shown that the left-hand side of (2.20) is equal to

$$\frac{\int_t^\infty\sum_{i=1}^n i\overline{F}^{i-1}(t)\left(\overline{F}(t)-\overline{F}(x)\right)^{n-i-1}\overline{F}^2(x)dx}{\overline{F}^n(t)}, \tag{2.21}$$

which is always non negative, leading us to inequality in (2.19). On the other hand, using Theorem 2.4 we have $M_{(n)}^r\leq M_{(n)}^1(t)$, for $r=1,2,\ldots,n$, implying that $M_{(n)}^r(t)\leq nM_1(t)-(n-1)M_2(t)$. This completes the proof of the theorem.

Remark 2.6. In Theorem 2.5, in the case where $r=1$, we can also obtain a lower bound for the MRL function of the system. In other words, we can show that

$$M_{(n)}^1(t)\geq nM_1(t)-\binom{n}{2}M_2(t).$$

To prove this, we have to show that

$$\sum_{i=3}^{n}(-1)^{i+1}\binom{n}{i}M_i(t)\geq 0. \tag{2.22}$$

It can be shown that the left-hand side of (2.22) is equal to

$$\frac{\int_t^\infty\left(\sum_{i=3}^n\overline{F}^i(t)\left(\overline{F}(t)-\overline{F}(x)\right)^{n-3-i}\sum_{j=0}^{i+1}j\right)\overline{F}^3(x)dx}{\overline{F}^n(t)},$$

which is always non negative. Hence, we have the lower bound as claimed.

Remark 2.7. It is worth noting that the bounds given in Theorem 2.5 and Remark 2.6 are analogs of the bounds given for the probability of occurrence of at least one of A_j in n events $A_1,\ldots,A_n$ of the sample space. It is known that (see for example, David and Nagaraja, 2003, p. 127) that for n events $A_1,\ldots,A_n$ for which $P(A_i)=P(A_1)$ and $P(A_iA_j)=P(A_1A_2)$ for all $i,j,\ i\neq j$,

$$nP(A_1)-\binom{n}{2}P(A_1A_2)\leq P\left(\bigcup_{i=1}^n A_i\right)\leq nP(A_1)-(n-1)P(A_1A_2).$$

Acknowledgments

The authors would like to thank the referee and the editor for valuable comments which resulted in the improvement of the presentation of this article. The research was done when the first author was visiting the Department of Statistics, Dokuz Eylul University, Izmir, Turkey.

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