The Mean Residual Lifetime and Mean Past Lifetime of k-out-of-n Structures at the System Level

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Abstract

In this paper we summarize some recent results on mean residual lifetime (MRL) and the mean past lifetime (MPL) of a k-out-of-n structure at the system level. In section 2, it is assumed that in a k-out-of-n system all the components of the system are working at time t. Under this condition, the MRL of the system is defined and several properties of that are derived. In section 3, under the assumption that system has failed at time t the MPL of the components of the system is defined and some of its properties are given.

Keywords and Phrases: Parallel system, hazard rate, reversed hazard rate, order statistics.

1 Introduction

The $(n-k+1)$-out-of-n structures are common structure of redundancy to improve the reliability of systems. A $(n-k+1)$-out-of-n system is a system consisting of n components which functions if an only if at least $(n-k+1)$ components of n components function. In special case when $k=n$ the system is known as the parallel system and in the case of $k=1$ the system is known as the series system. Let $T_1, ..., T_n$ denote the lifetimes of n components connected in a system with $(n-k+1)$-out-of-n structure. Assume that T_i's are independent and identically distributed random variables with common continuous distribution function F and survival function (reliability function) $\bar{F} = 1 - F$. Let also $T_{1:n} \leq T_{2:n} \leq \cdots \leq T_{n:n}$ be the ordered lifetimes of the components. Then the lifetime of a $(n-k+1)$-out-of-n system is $T_{k:n}$, $k = 1, 2..., n$. If, we denote the survival function of the system, at time t, by $\bar{S}(t)$ then it can be shown that

$$\begin{aligned} \bar{S}(t) &= P(T_{k:n} \geq t) && (1) \\ &= \sum_{i=0}^{k-1} \binom{n}{k} F^k(t) \bar{F}^{n-k}(t), \quad t > 0. && (2) \end{aligned}$$

In reliability theory and survival analysis to study the lifetime characteristics of an alive organism there have been defined several measures such as the mean residual lifetime (MRL) and the mean past lifetime (MPL). Assuming that each component of the system

has survived up to time t, the survival function of $T_i - t$, the residual lifetime of the components, given that $T_i > t$, $i = 1, ..., n$, is given by

$$\bar{F}(x|t) = \frac{\bar{F}(t+x)}{\bar{F}(t)}. \tag{3}$$

This is the corresponding conditional survival function of the components at age t. From (3), we get that the MRL m of each component is equal to

$$m(t) = E(T_i - t|T_i \geq t) = \int_0^\infty \bar{F}(x|t)dx = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)}, \qquad i = 1, 2, ..., n$$

It is well known that the MRL function m characterizes the distribution function F uniquely (see, for example, Kotz and Shanbhag (1980)).

The MPL corresponds to the mean time elapsed since the failure of T_i given that $T_i \leq t$. In this case , the random variable of interest is $t - T_i|T_i \leq t$. This conditional random variable shows the time elapsed since the failure of T_i given that it failed at or before t. The expectation of this random variable which we denote it by $k(t)$ is known as the MPL.

$$\begin{aligned} k(t) &= E(t - T|T \leq t) \\ &= \frac{\int_0^t F(x)dx}{F(t)} \end{aligned}$$

The MPL $k(t)$ is also characterizes the underlying distribution uniquely (see, for example Finkelstein (2002)).

Based on the above definitions of the MRL and MPL, as the lifetime of a $(n-k+1)-$out-of-n system is $T_{k:n}$, the MRL and MPL of the system are respectively equal to

$$E(T_{k:n} - t|T_{k:n} \geq t) = \frac{\int_t^\infty \bar{F}_{k:n}(x)dx}{\bar{F}_{k:n}(t)},$$

$$E(t - T_{k:n}|T_{k:n} \leq t) = \frac{\int_0^t F_{k:n}(x)dx}{F_{k:n}(t)},$$

where $F_{k:n}(t)$ and $\bar{F}_{k:n}$ denote the distribution function and survival function of $T_{k:n}$, respectively. Recently, some new concepts for the MRL and MPL of the system, at the system level, have been proposed in literature. Bairamov et al (2002) under the condition that none of the components of the system fails at time t, defined the MRL function of a parallel system, at the system level, as

$$M^1_{(n)}(t) = E(T_{n:n} - t|T_{1:n} \geq t),$$

and obtained several properties of it. Asadi and Bayramoglu (2005) have extended the above definition to the case that $n-k+1$, $k = 1, ..., n$ components of the system are working and the other components have already failed. In fact, they defined the MRL of a parallel system as

$$M^k_n(t) = E(T_{n:n} - t|T_{k:n} \geq t), \qquad k = 1, 2, ..., n.$$

Asadi and Bayramoglu (2006) have considered a $(n-k+1)$-out-of-n system and, under the condition that all of the components of the system are working, defined the MRL of the system, at the system level, as

$$H^k_n(t) = E(T_{k:n} - t|T_{1:n} > t).$$

The concept of the MPL of the components of a parallel system at the system level, under the condition that the system has failed at time or before time t, is introduced by Asadi (2006) as follows

$$L_n^r(t) = E(t - T_{r:n}|T_{n:n} \leq t).$$

This concept is extended to the $(n-k+1)$-out-of-n system by Tavangar and Asadi (2006) and several properties of it are explored. The aim of the present paper is to summarize the results on the MRL and the MPL of the system at the system level. In Section 2, we give the properties and results on the MRL at the system level. Section 3, deals with the results on the MPL at the system level.

2 The Mean Residual Lifetime of k-out-of-n system

Consider a parallel system consisting on n components. Assume that the components of the system have lifetimes $T_1, T_2, ..., T_n$ which are independent and identically distributed with a common distribution function F, and survival function $\bar{F} = 1 - F$. Bairamov et. al. (2002), under the condition that none of the components of the system fails at time t, defined the MRL function of a parallel system as

$$M_{(n)}^1(t) = E(T_{n:n} - t|T_{1:n} > t).$$

They have shown, among other things that, under some regularity conditions, the survival function $\bar{F}$ can be represented as

$$\bar{F}(t) = \exp\left\{-\frac{1}{n}\int_0^t \frac{1 + \frac{d}{dx}M_{(n)}^1(x)}{M_{(n)}^1(x) - M_{(n-1)}^1(x)}dx\right\}, \quad t > 0. \tag{4}$$

where $M_{(n-1)}^1(t)$ is the MRL of a parallel system having $n-1$ components. Asadi and Bairamov (2005) have given an extension of the $M_{(n)}^1(t)$ as follows:

$$M_{(n)}^r(t) = E(T_{n:n} - t|T_{r:n} > t), \qquad r = 1, 2, ..., n. \tag{5}$$

The $M_{(n)}^r(t)$ defined here is in fact the MRL of the system under the condition that $n - r + 1$, $r = 1, ..., n$ components of the system are working and the other components have already failed. It can be shown that, for t such that $\bar{F}(t) > 0$, $M_{(n)}^r(t)$ is given by

$$M_{(n)}^r(t) = \frac{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)M_{(n-i)}^1(t)}{\sum_{i=0}^{r-1}\binom{n}{i}\phi^i(t)} \tag{6}$$

where

$$M_{(k)}^1(t) = E(T_{k:k} - t|T_{1:k} > t) = \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}M_i(t), \qquad k = 1, 2, ..., \tag{7}$$

and
$M_j(t) = \frac{\int_t^\infty \bar{F}^j(x)dx}{\bar{F}^j(t)}$, $\phi(t) = \frac{F(t)}{\bar{F}(t)}$. An alternative form of writing $M_{(n)}^r(t)$ in terms of $M_{(k)}^1(t)$ is as follows:

$$M_{(n)}^r(t) = \sum_{m=0}^{r-1} p_m^n(t)M_{(n-m)}^1(t) \tag{8}$$

where $p_m^n(t) = P(Z_t = m|Z_t \leq k)$, $m = 0, ..., r-1$, and Z_t is distributed as binomial with parameters $(n, F(t))$. This result indicates that the MRL $M_{(n)}^r(t)$ is a mixtures of $M_{(n-i)}^1(t)$, for $i = 0, ..., r-1$. Asadi and Bayramoglu (2005) have studied the behaviour of $M_{(n)}^r(t)$ in terms of the behaviour of n and r. They showed that $M_{(n-1)}^1(t)$ is an increasing function of n. This shows that when the number of components of system increases, one would expect more lifetime for the system in average. However, the $M_{(n)}^r(t)$ is a decreasing function of r. That is, in a system where $(n-r+1)$ components are working we expect more lifetime, in average, than where $(n-r)$ components of the system are working.

Example 1 *Let $T_1, ..., T_n$ denote the lifetimes of the components of a parallel system . If $T_i^{'}s$ are distributed as exponential with mean λ respectively, then the MRL $M_{(n)}^r(t)$ is a non-increasing function of t. In fact it can be shown, in this case, that $M_{(n)}^r(t)$ has the following form*

$$\begin{aligned} M_{(n)}^r(t) &= \frac{\sum_{i=0}^{r-1}\binom{n}{i}(e^{\lambda t}-1)^i \sum_{j=1}^{n-i}(-1)^{j+1}\binom{n-i}{j}\frac{1}{\lambda j}}{\sum_{i=0}^{r-1}\binom{n}{i}(e^{\lambda t-1})^i} \\ &= \frac{\sum_{i=0}^{r-1}\binom{n}{i}(e^{\lambda t}-1)^i \sum_{j=1}^{n-i}\frac{1}{\lambda j}}{\sum_{i=0}^{r-1}\binom{n}{i}(e^{\lambda t}-1)^i}. \end{aligned} \tag{9}$$

Figure 1, shows the MRL a system containing 5 components where the components are assumed to have an exponential distribution with mean 1. The graph shows that when r increases the MRL of the system decreases and also the MRL is a non-increasing function of t.

In reliability theory the concept of increasing (decreasing) failure rate (IFR)(DFR) and the concept of decreasing (increasing) mean residual life (DMRL)(IMRL) are defined and widely studied to model the lifetime properties of a random variable with distribution function F. An absolutely continuous distribution function F is said to belong to class of IFR distributions if the corresponding hazard rate $r(t) = \frac{f(t)}{\bar{F}(t)}$ is non-decreasing in t. A continuous distribution function F is said to belong to DMRL distributions if the corresponding mean residual life $m(t)$ is a non-increasing function of t. We refer the reader to Barlow and Proschan (1981) for more details on these concepts. Asadi and Bayramoglu (2005) proved that when the components of a parallel system are independent and have identical distribution function F which is IFR then the distribution function corresponding to the lifetime of the system, i.e. the distribution of $T_{n:n} - t|T_{k:n} \geq t$, $k = 1, 2, \ldots, n$, is DMRL. In other words, when the components are IFR the MRL $M_{(n)}^r(t)$ is a decreasing function of time.

Asadi and Bayramoglu (2006) extended the results for parallel systems to the k-out-of-n system under the condition that at time t all the components of the system are operating. Consider a $(n-k+1)$-out-of-n system and assume the components of the system have independent lifetimes with common distribution function F and survival function (reliability function) $\bar{F} = 1 - F$. We assume that at time t, $t > 0$, all the components of the system are working, i.e. $T_{1:n} > t$. Therefore, the residual life time of the system is $T_{k:n} - t|T_{1:n} > t$. If we denote the MRL function of the system by $H_n^k(t)$

then it can be shown

$$\begin{aligned} H_n^k(t) &= E(T_{k:n} - t | T_{1:n} > t) \\ &= \sum_{s=0}^{k-1} \binom{n}{s} \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{n-s} \left(1 - \frac{\bar{F}(x)}{\bar{F}(t)}\right)^s dx \end{aligned} \tag{10}$$

The MRL life $H_n^k(t)$ defined above is in fact the MRL of $T_{k:n}$ at the system level. The MRL $H_n^r(t)$ can also be presented as

$$H_n^k(t) = \int_t^\infty P(Y_x^t \leq k-1) dx \tag{11}$$

where Y_x^t is a binomial random variable with probability of success $\frac{\bar{F}(x)}{\bar{F}(t)}$, $x < t$. It can also be easily seen, using Equation (10), that an alternative representation for the MRL function of the system is

$$H_n^k(t) = \sum_{s=0}^{k-1} \sum_{j=0}^{s} \binom{n}{s} \binom{s}{j} (-1)^j M_{n-s+j}(t) \tag{12}$$

where

$$M_{n-s+j}(t) = \frac{\int_t^\infty \bar{F}^{n-s+j}(x) dx}{\bar{F}^{n-s+j}(t)} \tag{13}$$

and denotes the MRL function of a series system consisting of $n - s + j$ components, $j = 0, ..., s$, $s = 0, ..., k-1$.

Example 2.1 *Let X be distributed as generalized Pareto distributions (GPD) with survival function*

$$\bar{F}(x) = (\frac{b}{ax+b})^{\frac{1}{a}+1}, \quad x > 0, \ b > 0, \ -1 < a. \tag{14}$$

(The GPD, as a family of distributions, includes the exponential distribution (when $a \to 0$), Pareto distribution (for $a > 0$) and the power distribution (for $-1 < a < 0$)). In this case, we have

$$M_{n-s+j}(t) = \frac{\int_t^\infty (ax+b)^{-(1/a+1)(n-s+j)} dx}{(at+b)^{-(1/a+1)(n-s+j)}} = \frac{at+b}{(a+1)(n-s+j) - a} \tag{15}$$

Hence, the MRL function of the system is given by

$$H_n^k(t) = \sum_{s=0}^{k-1} \binom{n}{s} \sum_{j=0}^{s} \binom{s}{j} (-1)^j \frac{at+b}{(n-s+j)(a+1) - a}.$$

Note that as $a \to 0$, then $H_n^k(t) \to b \sum_{s=0}^{k-1} \frac{1}{n-s}$ i.e. the MRL of a system having independent exponential components does not depend on t.

An important property of the MRL function $H_n^k(t)$ is that it characterizes the underlying distribution F uniquely. The following theorem shows that in the case where the distribution function F is absolutely continuous then it can be uniquely recovered by $H_n^r(t)$ and $H_{n-1}^{r-1}(t)$.

Theorem 2.2 *Let the components of the system have a common absolutely continuous distribution function F. Let also f and $\bar{F}$ denote the density and survival functions corresponding to F, respectively. Then the survival function $\bar{F}$ can be represented in terms of $H_n^k(t)$ and H_{n-1}^{k-1} as follows:*

$$\bar{F}(t) = e^{-\frac{1}{n}\int_0^t \eta(x)dx}, \qquad t > 0, \tag{16}$$

where

$$\eta(x) = \frac{1 + \frac{dH_n^k(x)}{dx}}{H_n^k(x) - H_{n-1}^{k-1}(x)}, \quad k = 1, ..., n,$$

and we define $H_{n-1}^0(x) = 0$.

Similar to the result obtained for parallel system when the hazard rate of the components of the system have monotone behaviour then the MRL of the system is monotone. In fact, it can be proved that when the components of the system have an IFR (DFR) distribution function F, then $H_n^k(t)$ is decreasing (increasing)in t. The following example gives an application of above result.

Example 2.3 *Let the components of the system have Weibull distribution with survival function*

$$\bar{F}(t) = e^{-(\frac{t}{\beta})^\alpha}, \quad t > 0,\ \alpha > 0,\ \beta > 0. \tag{17}$$

It is easy to see that the Weibull distribution with shape parameter $\alpha > 1$ has increasing hazard rate and with shape parameter $\alpha < 1$ has decreasing hazard rate. This implies, using the above result, that the MRL $M_n^r(t)$ of the system is decreasing for $\alpha > 1$ and is increasing for $\alpha < 1$.

The following theorem gives an ordering for the MRL's two $(n-k+1)$-out-of-n systems in the case where the components of the system are ordered in terms of hazard rates (Asadi and Bayramoglu (2006)).

Theorem 2.4 *Let S_1 and S_2 be two (n-k+1)-out-of-n systems with independent components. Let the components of S_1 (S_2) have the distribution function F(G), survival functions $\bar{F}$($\bar{G}$) and hazard rates r_F(r_G), respectively. If, for $t > 0$, $r_F(t) \leq r_G(t)$ then $H_n^{1k}(t) \geq H_n^{2k}(t)$, where H_n^{1k} and H_n^{2k} denote the mean residual lives of S_1 and S_2, respectively.*

3 The Mean Past Lifetime of $(n-k+1)$out-of-n system

Let us consider a parallel system with n non-negative independent components having a common continuous distribution function F. In real life situation, where systems often are not monitored continuously, one might be interested in getting inference more about the history of the system, e.g. when the individual components have failed. It might be important for engineers and system designers to have some information about the average time elapsed since the failure of those components. On the basis of the structure of parallel systems, when a component with lifetime $T_{r:n}$, $r = 1, 2, ..., n-1$, fails the system is continuing to operate until $T_{n:n}$ fails. In fact, the system can be considered as a black box in the sense that the exact failure time of $T_{r:n}$ in it is unknown. Motivated by this,

we assume that at time t the system is not working and in fact it has failed at time t or sometime before time t. Consider the conditional random variable $t-T_{r:n}|T_{n:n} \leq t$, $t > 0$, $r = 1, ..., n$,. This random variable shows, in fact, the time that has passed from the failure of the component with lifetime $T_{r:n}$ in the system given that the system has failed at or before time t. If we denote the expectation of this conditional random variable by $L_n^r(t)$, then, $L_n^r(t)$ measures the mean past lifetime (MPL) from the failure of the component with lifetime $T_{r:n}$ given that the system has lifetime less than or equal to t. Asadi (2006) has shown that for $r = 1, 2, ..., n$

$$\begin{aligned} L_n^r(t) &= E(t - T_{r:n}|T_{n:n} \leq t) \\ &= \sum_{i=r}^{n} \binom{n}{i} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} L_{i+j}(t). \end{aligned} \tag{18}$$

where for $k = 1, 2, ...$ $L_k(t)$ is given by

$$\begin{aligned} L_k(t) &= \frac{\int_0^t F^k(t-x)dx}{F^k(t)} \\ &= \frac{\int_0^t F^k(u)du}{F^k(t)}. \end{aligned} \tag{19}$$

Note that $L_k(t)$ is in fact $E(t - T_{k:k}|T_{k:k} \leq t)$ $k = 1, 2, \ldots,$ which is denoting the mean past time since the failure time of $T_{k:k}$ given that $T_{k:k} \leq t$, i.e. the MPL of $T_{k:k}$ in the component level. This shows that the MPL of the components of the system, in system level, is a combination of the MPL's of $T_{k:k}$, $k = r, r+1, ..., n$ in component level. It is easy to see that $L_n^r(t)$ can also be represented as

$$L_n^r(t) = \int_a^t P(Y_t^x \geq r)dx. \tag{20}$$

where Y_t^x is a random variable distributed as $Binomial(n, \theta_t(x))$, with $\theta_t(x) = \frac{F(x)}{F(t)}$, $x < t$.

It can be shown that $L_n^r(t)$ is a decreasing function of r, $r = 1, ..., n$ and is an increasing function of n.

Example 3.1 *Let $T_i^{'}s$ $i = 1, ..., n$ $n \geq 1$, be independent and suppose that they are distributed as uniform on $(0,\ \theta)$, $\theta > 0$. Then it can be shown that $L_n^r(t) = \frac{n-r+1}{n+1}t$, which is a linear function of t.*

An important property of the MPL $L_n^r(t)$ is that it characterizes the underlying distribution F uniquely. In fact, the distribution function F can be represented by $L_n^r(t)$ and $L_{n-1}^r(t)$. this is given in the following theorem (Asadi (2006)).

Theorem 3.2 *Let the components of the system have a common absolutely continuous strictly increasing distribution function F. Let also f denote the density function of F. Then the distribution function F can be represented in terms of $L_n^r(t)$ and $L_{n-1}^r(t)$ as follows:*

$$F(t) = \exp\{-\frac{1}{n}\int_t^b \frac{1 - \frac{dL_n^r(x)}{dx}}{L_n^r(x) - L_{n-1}^r(x)}dx\} \qquad t \in (a, b), \qquad n \geq 1,\ r = 1, 2..., n,, \tag{21}$$

with $M_0^r(t) = 0$ for $n = 1$.

Tavangar and Asadi (2006) has extended the concept of MPL to systems with k-out-of-n structure. Consider a $(n-k+1)$-out-of-n system consisting of n components. Assume again that the components of the system have lifetimes $T_1, T_2, ..., T_n$, respectively, where T_i's are random variables with a common continuous distribution function F and the reliability function $\bar{F}$, respectively . Suppose that the components of the system are starting to work at time $t = 0$ and the system fails at t or sometime before time t, $t > 0$. On noting that the lifetime of a $(n-k+1)$-out-of-n system is $T_{k:n}$, we have $T_{k:n} \leq t$. On the basis of the structure of the system if the failure time of the components are not monitored continuously then the exact failure time of components of the system with lifetimes $T_{1:n}, ..., T_{k-1:n}$ are unknown. The time that has elapsed from the failure of $T_{r:n}$, $r = 1, ..., k-1$ given that the system has failed at or before t is $t - T_{r:n}|T_{k:n} \leq t$. We denote the expectation of this by $L_n^{r,k}(t)$ and call it the MPL of the components of the system. That is

$$L_n^{r,k}(t) = E\left(t - T_{r:n}|T_{k:n} \leq t\right).$$

Note that in the case of $k = n$ the system becomes a parallel system and we have $L_n^{r,k}(t) = L_n^r(t)$.

Several properties of $L_n^{r,k}(t)$ is obtained by Tavangar and Asadi (2006). It can shown that the MPL $L_n^{r,k}(t)$ has the following relation with MPL $L_n^r(t)$.

$$\begin{aligned} L_n^{r,k}(t) &= E(t - T_{r:n}|T_{k:n} \leq t) \\ &= \frac{\sum_{m=k}^{n} \binom{n}{m} u_t^m L_m^r(t)}{\sum_{m=k}^{n} \binom{n}{m} u_t^m}, \end{aligned} \tag{22}$$

where $u_t = \frac{F(t)}{\bar{F}(t)}$ and

$$L_m^r(t) = E(t - T_{r:m}|T_{m:m} \leq t)$$

denotes the MPL of $T_{r:m}$, $r = 1, ..., m$, in a parallel system consisting of m components. The above representation can also be written as follows

$$L_n^{r,k}(t) = \sum_{m=k}^{n} p_m^n(t) S_m^r(t) \tag{23}$$

where $p_m^n(t) = P(Z_t = m|Z_t \geq k)$, $m = k, ..., n$, and Z_t is distributed as binomial with parameters $(n, F(t))$. This shows that the MPL of $T_{r:n}$, $r \leq k$, in a $(n-k+1)$-out-of-n system can be represented as the mixture of the MPL's of $T_{r:k}$, in $n-k+1$ parallel systems.

Remark 3.3 *The MPL $L_n^{r,k}(t)$ (and hence the MPL $L_n^r(t)$) is bounded above by t. It follows from the fact that for all $t > 0$, $E(T_{r:n}|T_{k:n} \leq t) \geq 0$.*

Example 3.4 *Let T_i's, $i = 1, 2, ..., n$, be independent and assume that they are distributed as exponential with mean 1. One can show that for $r = 1, ..., k$ and $1 \leq k \leq n$,*

$$L_n^{r,k}(t) = \frac{\sum_{m=k}^{n} \binom{n}{m} (e^t - 1)^m M_m^r(t)}{\sum_{m=k}^{n} \binom{n}{m} (e^t - 1)^m}$$

where

$$L_m^r(t) = \sum_{i=r}^{m} \binom{m}{i} \sum_{j=0}^{m-i} (-1)^j \binom{m-i}{j} \frac{t + \sum_{s=1}^{i+j} (-1)^s \binom{i+j}{s} \frac{1}{s}(1 - e^{-st})}{(1 - e^{-t})^{i+j}}.$$

Figures 1 and 2 show the graphs of MPL $L_n^{r,k}(t)$ of exponential distribution for the case where $n = 5,\ k = 3$ and $n = 5,\ k = 4$.

Figure 1: Figure 2:

Example 3.5 *Let T_i's, $i = 1, 2, ..., n$, be independent random variables with density function*

$$f(x) = \begin{cases} \frac{3}{2}(1-x)^2 & 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be shown that the MPL of the components are given as follows:

$$L_n^{r,k}(t) = \frac{\sum_{m=k}^{n} \binom{n}{m} \left(\frac{1-(1-t)^3}{1+(1-t)^3}\right)^m L_m^r(t)}{\sum_{m=k}^{n} \binom{n}{m} \left(\frac{1-(1-t)^3}{1+(1-t)^3}\right)^m},$$

where

$$L_m^r(t) = \sum_{i=r}^{m} \binom{m}{i} \sum_{j=0}^{m-i} (-1)^j \binom{m-i}{j} \frac{t + \sum_{s=1}^{i+j} (-1)^s \binom{i+j}{s} \frac{1}{3s+1} \{1-(1-t)^{3s+1}\}}{\{1-(1-t)^3\}^{i+j}}.$$

Figures 3 and 4 show the graph of MPL $L_n^{r,k}(t)$ for $n = 5,\ k = 3$ and $n = 5,\ k = 4$.

Figure 3: Figure 4:

The following theorem shows several monotonicity properties of $L_n^{r,k}(t)$.

Theorem 3.6 *Let $L_n^{r,k}(t)$ be the MPL of a $(n-k+1)$-out-of-n system. Then*

- *For fixed values of k and n, $L_n^{r,k}(t)$ is a decreasing function of r, $r = 1, 2, ..., k$.*
- *For fixed n and r, $L_n^{r,k}(t)$ is increasing in k, $k = r, ..., n$.*
- *For fixed values of r and k, $L_n^{r,k}(t)$ is an increasing function of n.*

The next two theorems show some results on the behaviour of the MPL with respect to the behaviour of the reversed hazard rate of the underlying distribution function F.

Theorem 3.7 *Let $r(t)$, the reversed hazard rate of the components of the system, be decreasing in t, $0 < t$.Then $L_n^{r,k}(t)$ is an increasing function of t.*

Theorem 3.8 *Consider two $(n-k+1)$-out-of-n systems $\mathcal{S}_1$ and $\mathcal{S}_2$ each consisting of n independent and identical components. Let the components of $\mathcal{S}_1(\mathcal{S}_2)$ have the common distribution $F(G)$ (with the same support), and reversed hazard functions $r_F(r_G)$, respectively. If $r_F(t) \geq r_G(t)$, $t > 0$, then*

$$L_n^{r,k}(t) \leq K_n^{r,k}(t),\ r = 1, 2, ..., k$$

where we assume $L_n^{r,k}(t)$ *is the MPL* $\mathcal{S}_1$ *and is given by (22) and* $K_n^{r,k}(t)$ *is the MPL of* $\mathcal{S}_2$ *and has the following representation*

$$K_n^{r,k}(t) = \frac{\sum_{m=k}^{n} \binom{n}{m} v_t^m Z_m^r(t)}{\sum_{m=k}^{n} \binom{n}{m} v_t^m}$$

where $v_t = \frac{G(t)}{\bar{G}(t)}$ *and*

$$Z_m^r(t) = \sum_{i=r}^{m} \binom{m}{i} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} Z_{i+j}(t). \tag{24}$$

where

$$Z_k(t) \quad = \quad \frac{\int_0^t G^k(u)du}{G^k(t)}. \tag{25}$$

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