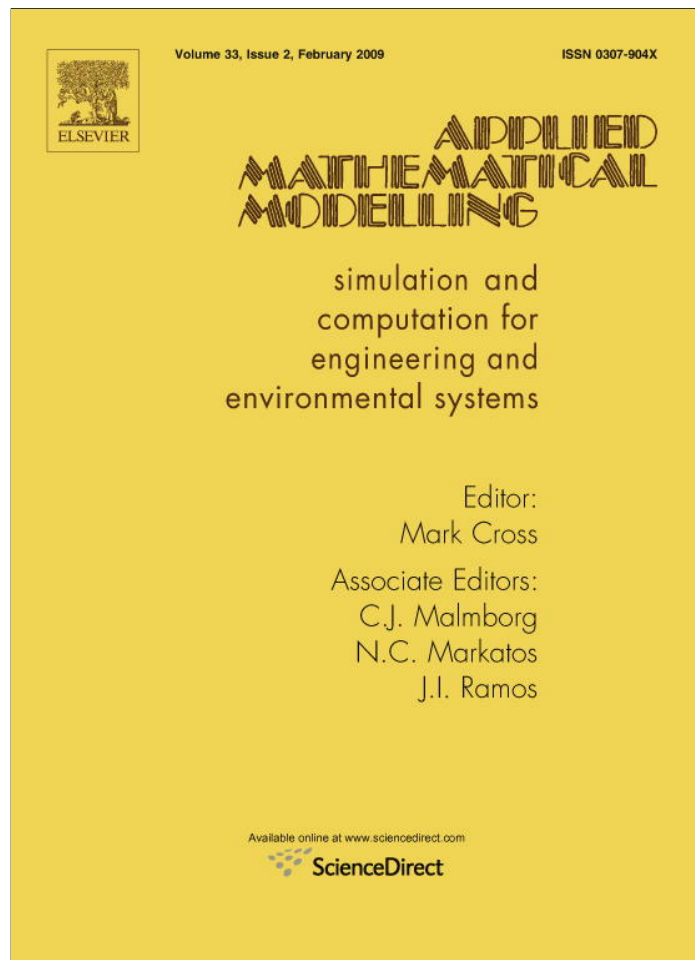


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# Parallel and $k$ -out-of- $n$ : $G$ systems with nonidentical components and their mean residual life functions

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## Abstract

A system with  $n$  independent components which has a  $k$ -out-of- $n$ :  $G$  structure operates if at least  $k$  components operate. Parallel systems are 1-out-of- $n$ :  $G$  systems, that is, the system goes out of service when all of its components fail. This paper investigates the mean residual life function of systems with independent and nonidentically distributed components. Some examples related to some lifetime distribution functions are given. We present a numerical example for evaluating the relationship between the mean residual life of the  $k$ -out-of- $n$ :  $G$  system and that of its components.

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*Keywords:*  $k$ -Out-of- $n$ :  $G$  systems; Parallel systems; Mean residual life function; Symmetric functions; Permanents

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## 1. Introduction

A technical system has a  $k$ -out-of- $n$ :  $G$  structure if at least  $k$  of the  $n$  components work (or are good) for the entire system and fails if  $n - k + 1$  or more components fail. Hence, a  $k$ -out-of- $n$ :  $G$  system breaks down at the time of the failure of the  $(n - k + 1)$ th component. As all components start working at the same time, this approach leads to an active redundancy of  $n - k$  components. Important particular cases of  $k$ -out-of- $n$ :  $G$  systems are parallel and series systems corresponding to  $k = 1$  and  $k = n$ , respectively.

Assuming that each component of the system has survived up to time  $t$ , the survival function of  $X_i - t$  given that  $X_i > t$ ,  $i = 1, \dots, n$ , is

$$S_i(x|t) = \frac{\bar{F}_i(t+x)}{\bar{F}_i(t)}. \quad (1)$$

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**Nomenclature**

*Acronyms*

- MRL mean residual life
- IID independent and identically distributed
- INID independent and nonidentically distributed

*Notation*

- $\bar{F}_{(n)}(t)$  survival function of a system
- $\Psi_i(t)$  MRL function of the  $i$ th component
- $\Psi_{n:n}(t)$  MRL function of the  $n$  components parallel system
- $\Phi_{n:n}(t)$  MRL function at the system level of the  $n$  components parallel system
- $H_{(n)}^k(t)$  MRL function of the  $(n - k + 1)$ -out-of- $n$ :  $G$  system under the condition that all components are alive at time  $t$

The MRL function of each component is

$$\Psi_i(t) = E(X_i - t | X_i > t) = \int_0^\infty S_i(x|t) dx = \frac{1}{\bar{F}_i(t)} \int_t^\infty \bar{F}_i(x) dx, \quad i = 1, \dots, n. \tag{2}$$

Although there are many papers related to the  $k$ -out-of- $n$ :  $G$  system, few have focused on the MRL of parallel and  $k$ -out-of- $n$ :  $G$  systems. For the recent results on the MRL functions of  $k$ -out-of- $n$  systems one can see Khal-edi and Shaked [1], Navarro and Shaked [2], Navarro and Eryilmaz [3], Hu et al. [4]. Asadi and Bairamov [5,6] have defined the MRL functions of parallel and  $k$ -out-of- $n$ :  $G$  systems consisting of identical and independent components and have obtained some of their properties. Sarhan and Abouammoh [7] have investigated the reliability of nonrepairable  $k$ -out-of- $n$  systems with nonidentical components subjected to independent and common shocks. Li and Chen [8] studied the aging properties of the residual life length of a  $k$ -out-of- $n$  system with independent (not necessarily identical) components given that the  $(n - k)$ th failure has occurred at time  $t > 0$ . Most of the fault-tolerant systems such as parallel and  $k$ -out-of- $n$ :  $G$  systems, consist of nonidentical components. This type of structures find wide applications in both industrial and technical areas.

In this study, the MRL function for parallel and  $k$ -out-of- $n$ :  $G$  systems consisting of  $n$  INID components are investigated. Section 2 focuses on the MRL function of the  $k$ -out-of- $n$ :  $G$  system. In Section 3, a detailed coverage on MRL evaluation for parallel systems is provided. In Section 4, we provide an example on the MRL of the  $k$ -out-of- $n$ :  $G$  system with INID components demonstrating for selected parameters, to examine the relationship between the MRL of the system and its components.

*1.1. Preliminaries*

The MRL function for the nonidentical case can be expressed in terms of symmetric functions and permanents. We describe shortly below the definition of symmetric functions and permanents and provide some of their useful properties.

**Definition 1.** Let  $\mathbf{x} = (x_1, \dots, x_n) \in \mathfrak{R}^n$ , ( $n \geq 1$ ). The  $r$ th ( $1 \leq r \leq n$ ) elementary symmetric function denoted by  $\sigma_r(x_1, \dots, x_n)$ , is the sum of all products of  $r$  distinct variables chosen from  $n$  variables [9]. That is

$$\sigma_r(x_1, \dots, x_n) = \sigma_r(\mathbf{x}) = \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} \dots x_{i_r}.$$

It is convenient to define  $\sigma_r(\mathbf{x}) = 0$  for  $r < 0$  and  $r > n$ . And  $\sigma_r(\mathbf{x}) = 1$  when  $r = 0$ . The generating function  $G(x)$  for the elementary symmetric function is

$$G(x) = \prod_{i=1}^n (1 + x_i) = \sum_{r=0}^n (-1)^r \sigma_r(\mathbf{x}). \tag{3}$$

A recurrence relation for the elementary symmetric functions can be obtained from (3) as [10]

$$\sigma_r(x_1, \dots, x_n) = \sigma_r(x_1, \dots, x_{n-1}) + x_n \sigma_{r-1}(x_1, \dots, x_{n-1}). \quad (4)$$

When the variables are independent but not assumed to be identically distributed, the distributions of order statistics can be expressed in terms of permanents. Consider  $S_n$  as the set of permutations of  $1, 2, \dots, n$ . For an  $n \times n$  matrix, say  $A = (a_{i,j})$ , the permanent of  $A$ , denoted by  $PerA$ , can be defined by

$$PerA = \sum_{\pi \in S_n} \prod_{i=1}^n a_{i,\pi(i)},$$

where  $\pi = (\pi(1), \pi(2), \dots, \pi(n))$ . If  $a_1, a_2, \dots$ , are column vectors, then

$$\left[ \underbrace{a_1}_{i_1}, \underbrace{a_2}_{i_2}, \dots \right] = [a_1 \mathbf{1}'_{i_1}, a_2 \mathbf{1}'_{i_2}, \dots],$$

where  $\mathbf{1}'_{i_1}$  represents a column vector of 1's of length  $i_1$ . The permanent is defined just like the determinant, except that all signs in the expansion are positive. Also the permanent states a Laplace expansion along any row or column of the matrix. If we denote by  $A(i, j)$  the matrix obtained by deleting row  $i$  and column  $j$  of the  $n \times n$  matrix  $A$ , then for  $i, j = 1, 2, \dots, n$

$$PerA = \sum_{j=1}^n a_{ij} PerA(i, j) \quad \text{and} \quad PerA = \sum_{i=1}^n a_{ij} PerA(i, j).$$

Vaughan and Venables [11] have shown that the density of  $X_{r:n}$  is expressed in terms of permanents, when  $X_{1:n}, X_{2:n}, \dots, X_{n:n}$  are order statistics of the independent random variables with absolutely continuous distribution functions  $F_1, F_2, \dots, F_n$  and densities  $f_1, f_2, \dots, f_n$ , respectively. The distribution function of  $X_{r:n}$  ( $1 \leq r \leq n$ ) was given by Bapat and Beg [12] as

$$P(X_{r:n} \leq x) = \sum_{i=r}^n \frac{1}{i!(n-i)!} Per \begin{bmatrix} F_1(x) & 1 - F_1(x) \\ \vdots & \vdots \\ \underbrace{F_n(x)}_i & \underbrace{1 - F_n(x)}_{n-i} \end{bmatrix}, \quad -\infty < x < \infty,$$

where  $PerA$  denotes the permanent of a square matrix  $A$ . A simple argument shows [13] that

$$P(X_{r:n} \leq x) = \sum_{i=r}^n P(\text{exactly } i \text{ variables from } X_1, \dots, X_n \text{ are } \leq x) = \sum_{i=r}^n \sum_{j_1, \dots, j_n} \prod_{l=1}^i F_{j_l}(x) \prod_{l=i+1}^n [1 - F_{j_l}(x)],$$

where the summation extends over all permutations  $j_1, \dots, j_n$  of  $1, \dots, n$  for which  $j_1 < \dots < j_i$  and  $j_{i+1} < \dots < j_n$ .

## 2. The MRL function of the $k$ -out-of- $n$ : $G$ system

Let  $X_{k:n}$ ,  $k = 1, 2, \dots, n$ , represents the lifetime of  $(n - k + 1)$ -out-of- $n$ :  $G$  system. Considering the following definition, the MRL function of a  $k$ -out-of- $n$ :  $G$  system can be given as follows under the assumption that  $X_1, X_2, \dots, X_n$  are INID random variables with distribution function  $F_i$  and survival function  $\bar{F}_i = 1 - F_i$ . Let also  $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$  be the ordered lifetimes of the components.

**Definition 2.** The MRL function of the  $(n - k + 1)$ -out-of- $n$ :  $G$  system under the condition that all components alive at time  $t$ , is

$$H_{(n)}^k(t) = E(X_{k:n} - t | X_{1:n} > t), \quad k = 1, 2, \dots, n. \quad (5)$$

**Theorem 1.** If  $H_{(n)}^k(t)$  is the MRL of the parallel system defined as (5), then for  $\bar{F}(t) > 0$ ,

$$H_{(n)}^k(t) = \frac{\sum_{i=0}^{k-1} \binom{n}{i}}{n! \prod_{i=1}^n \bar{F}_i(t)} \int_t^\infty \text{Per} \left[ \underbrace{\bar{F}(t) - \bar{F}(x)}_i \underbrace{\bar{F}(x)}_{n-i} \right] dx, \quad k = 1, 2, \dots, n, \quad t > 0. \quad (6)$$

**Proof.** If  $S_{(n)}^k(x|t)$  denotes the survival function of conditional random variable  $X_{k:n} - t | X_{1:n} > t$  then for  $x > 0$ ,

$$\begin{aligned} S_{(n)}^k(x|t) &= P(X_{k:n} - t | X_{1:n} > t) = \frac{P(X_{k:n} > x + t, X_{1:n} > t)}{P(X_{1:n} > t)} \\ &= \frac{\sum_{i=0}^{k-1} \sum_{j_1, \dots, j_n} \prod_{l=1}^i [F_{j_l}(x + t) - F_{j_l}(t)] \prod_{l=i+1}^n \bar{F}_{j_l}(x + t)}{\prod_{i=1}^n \bar{F}_i(t)} \\ &= \frac{\sum_{i=0}^{k-1} \sum_{j_1, \dots, j_n} \prod_{l=1}^i [\bar{F}_{j_l}(t) - \bar{F}_{j_l}(x + t)] \prod_{l=i+1}^n \bar{F}_{j_l}(x + t)}{\prod_{i=1}^n \bar{F}_i(t)} \end{aligned} \quad (7)$$

Hence the full sum is recognizable as the permanent of a matrix, so  $S_{(n)}^k(x|t)$  has the following expression:

$$\frac{\sum_{i=0}^{k-1} \frac{1}{i!(n-i)!} \text{Per} \left[ \underbrace{\bar{F}(t) - \bar{F}(x + t)}_i \underbrace{\bar{F}(x + t)}_{n-i} \right]}{\prod_{i=1}^n \bar{F}_i(t)}. \quad (8)$$

Finally, given that all the components of the system are working at time  $t$ , the MRL function of the system is

$$H_{(n)}^k(t) = \int_0^\infty S(x|t) dx = \frac{\sum_{i=0}^{k-1} \binom{n}{i} \int_t^\infty \text{Per} \left[ \underbrace{\bar{F}(t) - \bar{F}(x)}_i \underbrace{\bar{F}(x)}_{n-i} \right] dx}{n! \prod_{i=1}^n \bar{F}_i(t)}, \quad k = 1, 2, \dots, n, \quad t > 0.$$

Thus the proof is completed.  $\square$

The motivation for this structure can be given as an example of the high priority freight train, which is structured as a 3-out-of-4:  $G$  system consisting of four locomotives [14]. The train is delayed only if two or more locomotives fail. It is assumed that the four locomotives in a train fail independently and times to failure for locomotives are distributed as exponential distributions. In this example,  $X_{2:4}$  represents the lifetime of this system. The MRL of the system is

$$H_{(4)}^2(t) = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_1 + \lambda_3 + \lambda_4} + \frac{1}{\lambda_1 + \lambda_2 + \lambda_4} + \frac{1}{\lambda_2 + \lambda_3 + \lambda_4} - \frac{3}{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}.$$

This expression can also be obtained from Remark 5 of Navarro and Hernandez [15]. It is clear that the MRL of the system is a decreasing function of failure rates  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  as expected.

### 3. The MRL function of a parallel system with INID components

A 1-out-of- $n$ :  $G$  system is a parallel system. In this section we investigate the MRL function of the parallel system with INID components. At time  $t, t > 0$ , the residual lifetime of the  $n$  components parallel system is  $X_{n:n} - t | X_{n:n} > t$ . If  $S_{n:n}(x|t)$  denotes the survival function of this conditional random variable then, it can be shown that, for  $x > 0$ :

$$\begin{aligned} S_{n:n}(x|t) &= P(X_{n:n} > x + t | X_{n:n} > t) \\ &= \frac{1}{1 - \prod_{i=1}^n F_i(t)} \left[ 1 - \prod_{i=1}^n F_i(x + t) \right]. \end{aligned}$$

The MRL function is

$$\Psi_{n:n}(t) = E(X_{n:n} - t | X_{n:n} > t) = \int_0^\infty S_{n:n}(x|t) dx = \frac{1}{1 - \prod_{i=1}^n F_i(t)} \int_t^\infty \left[ 1 - \prod_{i=1}^n F_i(x) \right] dx. \tag{9}$$

By using the generating function (3), we rewrite the product part of (9) as

$$\prod_{i=1}^n [1 - \bar{F}_i(t)] = \sum_{r=0}^n (-1)^r \sum_{1 \leq i_1 < \dots < i_r \leq n} \bar{F}_{i_1}(t) \dots \bar{F}_{i_r}(t) = \sum_{r=0}^n (-1)^r \sigma_r(\bar{F}_1(t), \dots, \bar{F}_n(t)).$$

Then by Definition 1  $\sigma_0(x_1, \dots, x_n) = 1$ , (9) can be expressed in terms of the elementary symmetric functions as follows:

$$\begin{aligned} \Psi_{n:n}(t) &= \frac{1}{1 - \sum_{r=0}^n (-1)^r \sigma_r(\bar{F}_1(t), \dots, \bar{F}_n(t))} \int_t^\infty \left[ 1 - \sum_{r=0}^n (-1)^r \sigma_r(\bar{F}_1(x), \dots, \bar{F}_n(x)) \right] dx, \\ \Psi_{n:n}(t) &= \frac{1}{\sum_{r=1}^n (-1)^{r+1} \sigma_r(\bar{F}_1(t), \dots, \bar{F}_n(t))} \sum_{r=1}^n (-1)^{r+1} \int_t^\infty \sigma_r(\bar{F}_1(x), \dots, \bar{F}_n(x)) dx. \end{aligned} \tag{10}$$

The following examples illustrate this concept.

**Example 1.** An important life distribution is the exponential distribution. Let

$$F_i(x) = 1 - e^{-\lambda_i x}, \quad x \geq 0, \quad \lambda_i > 0, \quad i = 1, \dots, n.$$

The MRL function of such a system containing three components has the following form:

$$\Psi_{3:3}(t) = \frac{\sum_{r=1}^3 (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq 3} \frac{1}{\sum_{k=1}^r \lambda_{i_k}} \prod_{k=1}^r e^{-\lambda_{i_k} t}}{\sum_{r=1}^3 (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq 3} \prod_{k=1}^r e^{-\lambda_{i_k} t}},$$

where the denominator is the survival function of the parallel system and can be written as

$$\begin{aligned} \bar{F}_{3:3}(t) &= \sum_{r=1}^3 (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq 3} \prod_{k=1}^r e^{-\lambda_{i_k} t} \\ &= \sum_{j=1}^3 e^{-\lambda_j t} - e^{-(\lambda_1 + \lambda_2)t} - e^{-(\lambda_1 + \lambda_3)t} - e^{-(\lambda_2 + \lambda_3)t} + e^{-(\lambda_1 + \lambda_2 + \lambda_3)t}. \end{aligned}$$

Note that  $\bar{F}_{3:3}$  is a generalized mixture of the survival functions of series systems. Navarro and Hernandez [15] showed that this is a general property of  $k$ -out-of- $n$  systems.

The MRL function of a system containing  $n$  exponential components has the form

$$\Psi_{n:n}(t) = \frac{\sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq n} \frac{1}{\sum_{k=1}^r \lambda_{i_k}} \prod_{k=1}^r e^{-\lambda_{i_k} t}}{\sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq n} \prod_{k=1}^r e^{-\lambda_{i_k} t}}. \tag{11}$$

For the large values of  $t$ , the MRL function is

$$\lim_{t \rightarrow \infty} \Psi_{n:n}(t) = \frac{1}{\min(\lambda_1, \lambda_2, \dots, \lambda_n)}. \tag{12}$$

That is, it is asymptotically equivalent to the MRL functions of its components. Indeed, assume that  $\lambda_1 = \min(\lambda_1, \lambda_2, \dots, \lambda_n)$ . Then we have from (11)

$$\Psi_{n:n}(t) = \frac{e^{-\lambda_1 t} \left[ \sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq n} \frac{1}{\sum_{k=1}^r \lambda_{i_k}} e^{(\lambda_1 - \sum_{k=1}^r \lambda_{i_k}) t} \right]}{e^{-\lambda_1 t} \left[ \sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq n} e^{(\lambda_1 - \sum_{k=1}^r \lambda_{i_k}) t} \right]}. \tag{13}$$

Since  $\lambda_1 - \sum_{k=1}^r \lambda_{i_k} < 0$  and  $\lim_{t \rightarrow \infty} e^{(\lambda_1 - \sum_{k=1}^r \lambda_{i_k}) t} = 0$ , (12) follows from (13).

Navarro and Hernandez [15] prove under some conditions that, (12) is a general property of the MRL functions of parallel systems.

For identically distributed components, i.e. when  $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$ , the MRL function of the system in (11) is reduced to expression given below:

$$\Psi_{n:n}(t) = \frac{\sum_{r=1}^n (-1)^{r+1} \binom{n}{r} \frac{1}{r\lambda} e^{-r\lambda t}}{\sum_{r=1}^n (-1)^{r+1} \binom{n}{r} e^{-r\lambda t}}. \tag{14}$$

**Example 2.** Another important class of life distributions is the power distribution. Let

$$F_i(x) = 1 - (1 - x)^{\theta_i}, \quad 0 < x < 1, \quad i = 1, \dots, n.$$

The MRL function of a parallel system containing  $n$  components has the following form:

$$\Psi_{n:n}(t) = \frac{(1-t) \sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq n} \frac{1}{1 + \sum_{k=1}^r \theta_{i_k}} \prod_{k=1}^r (1-t)^{\theta_{i_k}}}{\sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < \dots < i_r \leq n} \prod_{k=1}^r (1-t)^{\theta_{i_k}}}. \tag{15}$$

In the following theorem we present a recurrence formula for the MRL function of the parallel system given in (10).

**Lemma 1.** Let  $A_n(t) = 1 - \prod_{i=1}^n [1 - \bar{F}_i(t)]$  be the survival function of  $n$  INID components parallel system. Then

$$A_n(t) = A_{n-1}(t) + \bar{F}_n(t)[1 - A_{n-1}(t)].$$

**Proof.** Since the survival function of a parallel system consisting of  $(n - 1)$  INID components can be defined as  $A_{n-1}(t) = \sum_{r=1}^{n-1} (-1)^{r+1} \sigma_r(\bar{F}_1(t), \dots, \bar{F}_{n-1}(t))$ , using (4) one can write  $A_n(t)$  in terms of  $A_{n-1}(t)$  as follows:

$$\begin{aligned} A_n(t) &= \sum_{r=1}^n (-1)^{r+1} \sigma_r(\bar{F}_1(t), \dots, \bar{F}_n(t)) \\ &= \sum_{r=1}^n (-1)^{r+1} [\sigma_r(\bar{F}_1(t), \dots, \bar{F}_{n-1}(t)) + \bar{F}_n(t) \sigma_{r-1}(\bar{F}_1(t), \dots, \bar{F}_{n-1}(t))] \\ &= \sum_{r=1}^{n-1} (-1)^{r+1} \sigma_r(\bar{F}_1(t), \dots, \bar{F}_{n-1}(t)) + \bar{F}_n(t) \sum_{r=1}^n (-1)^{r+1} \sigma_{r-1}(\bar{F}_1(t), \dots, \bar{F}_{n-1}(t)). \quad \square \end{aligned}$$

**Theorem 2.** Let  $\Psi_{k:k}(t)$  be the MRL function of the parallel system with INID components having distribution function  $F_i, i = 1, 2, \dots, k$ . Then,

$$\Psi_{n:n}(t) = \omega(t) \Psi_{n-1:n-1}(t) + \delta(t), \quad t > 0, \tag{16}$$

where

$$\omega(t) = \frac{A_{n-1}(t)}{A_{n-1}(t) + \bar{F}_n(t)[1 - A_{n-1}(t)]}, \quad \delta(t) = \frac{\int_t^\infty \bar{F}_n(x)[1 - A_{n-1}(x)] dx}{A_{n-1}(t) + \bar{F}_n(t)[1 - A_{n-1}(t)]}$$

and

$$A_n(t) = \sum_{r=1}^n (-1)^{r+1} \sigma_r(\bar{F}_1(t), \dots, \bar{F}_n(t)). \tag{17}$$

**Proof.** For the MRL function of a system consisting of  $(n - 1)$  nonidentical components we have,

$$\sum_{r=1}^{n-1} (-1)^{r+1} \sigma_r(\bar{F}_1(t), \dots, \bar{F}_{n-1}(t)) \Psi_{n-1:n-1}(t) = \sum_{r=1}^{n-1} (-1)^{r+1} \int_t^\infty \sigma_r(\bar{F}_1(x), \dots, \bar{F}_{n-1}(x)) dx. \quad (18)$$

The MRL function of a system consisting of  $n$  nonidentical components is

$$\Psi_{n:n}(t) = \frac{1}{A_n(t)} \sum_{r=1}^n (-1)^{r+1} \int_t^\infty \sigma_r(\bar{F}_1(x), \dots, \bar{F}_n(x)) dx. \quad (19)$$

Now using Lemma 1, and (18), (19) follows that

$$\begin{aligned} \Psi_{n:n}(t) &= \frac{\sum_{r=1}^n (-1)^{r+1} \int_t^\infty [\sigma_r(\bar{F}_1(x), \dots, \bar{F}_{n-1}(x)) + \bar{F}_n(x) \sigma_{r-1}(\bar{F}_1(x), \dots, \bar{F}_{n-1}(x))] dx}{A_{n-1}(t) + \bar{F}_n(t)[1 - A_{n-1}(t)]} \\ &= \frac{(A_{n-1}(t) \Psi_{n-1:n-1}(t) + \sum_{r=0}^{n-1} (-1)^r \int_t^\infty \bar{F}_n(x) \sigma_r(\bar{F}_1, \dots, \bar{F}_{n-1}) dx)}{A_{n-1}(t) + \bar{F}_n(t)[1 - A_{n-1}(t)]}. \end{aligned}$$

Since it is easy to show that

$$\sum_{r=0}^{n-1} (-1)^r \sigma_r(\bar{F}_1(x), \dots, \bar{F}_{n-1}(x)) = 1 - A_{n-1}(x),$$

we obtain

$$\Psi_{n:n}(t) = \frac{A_{n-1}(t)}{A_{n-1}(t) + \bar{F}_n(t)[1 - A_{n-1}(t)]} \Psi_{n-1:n-1}(t) + \frac{\int_t^\infty \bar{F}_n(x)[1 - A_{n-1}(x)] dx}{A_{n-1}(t) + \bar{F}_n(t)[1 - A_{n-1}(t)]}.$$

Thus the theorem is proved.  $\square$

Recurrence relation (16) expresses  $\Psi_{n:n}(t)$  in terms of the MRL function and the survival function of  $n - 1$  INID components parallel system.

For the case of a system having IID lifetimes with distribution function  $F(x)$  an easy reduction follows from Theorem 2:

$$\Psi_{n:n}(t) = \frac{1 - F^{n-1}(t)}{1 - F^n(t)} \Psi_{n-1:n-1}(t) + \frac{\int_t^\infty \bar{F}(x) F^{n-1}(x) dx}{1 - F^n(t)}. \quad (20)$$

### 3.1. The MRL function of a parallel system having $n$ INID components all alive at time $t$

Consider a parallel system with INID components each following the distribution function  $F_i$  and survival function  $\bar{F}_i = 1 - F_i, i = 1, 2, \dots, n$ . When the system is put into operation at time  $t$ , all components are working. Let also  $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$  be the ordered lifetimes of the components. The consideration of the MRL function of this system leads us to the following definition.

**Definition 3.** The MRL function of a system under the condition all components are alive at time  $t$ , i.e.,  $X_{1:n} > t$ , is

$$\Phi_{n:n}(t) = E(X_{n:n} - t | X_{1:n} > t) = E(X_{n:n} | X_{1:n} > t) - t. \quad (21)$$

This function is called the MRL function at the system level in [5,6].

As it follows from the Theorem 1, we obtain the MRL function defined in (21) by substituting  $k = n$

$$\Phi_{n:n}(t) = \frac{\sum_{i=0}^{n-1} \binom{n}{i}}{n! \prod_{i=1}^n \bar{F}_i(t)} \int_t^\infty \text{Per} \left[ \underbrace{\bar{F}(t) - \bar{F}(x)}_i \underbrace{\bar{F}(x)}_{n-i} \right] dx, \quad t > 0. \quad (22)$$

In the following theorem, we give an alternative expression for (21).

**Theorem 3.** Let  $\Phi_{n:n}(t)$  be the MRL function of a system having a parallel structure and consisting of  $n$  INID components with distribution function  $F_i, i = 1, 2, \dots, n$ , respectively. Given that all components of the system are working at time  $t$  then

$$\Phi_{n:n}(t) = \frac{1}{\prod_{i=1}^n \bar{F}_i(t)} \frac{1}{(n-1)!} \int_t^\infty x \text{Per} \left[ \underbrace{f(x)}_1 \underbrace{F(x) - F(t)}_{n-1} \right] dx - t, \quad t > 0. \tag{23}$$

**Proof.** We have for  $x \geq t$

$$P(X_{n:n} \leq x | X_{1:n} > t) = \frac{\prod_{i=1}^n [F_i(x) - F_i(t)]}{\prod_{i=1}^n \bar{F}_i(t)}. \tag{24}$$

Differentiating (24) with respect to  $x$  we obtain the probability density function of the conditional random variable  $(X_{n:n} | X_{1:n} > t)$  as

$$\frac{1}{\prod_{i=1}^n \bar{F}_i(t)} \sum_{k=1}^n f_k(x) \prod_{i \neq k} [F_i(x) - F_i(t)]. \tag{25}$$

Using the identity (25) we have

$$\Phi_{n:n}(t) = \frac{1}{\prod_{i=1}^n \bar{F}_i(t)} \int_t^\infty x \sum_{k=1}^n f_k(x) \prod_{i \neq k} [F_i(x) - F_i(t)] dx - t.$$

It can be seen that

$$\sum_{k=1}^n f_k(x) \prod_{i \neq k} [F_i(x) - F_i(t)] = \sum_{j_1, \dots, j_n} f_{j_1}(x) \prod_{l=2}^n [F_{j_l}(x) - F_{j_l}(t)],$$

where the summation extends over all permutations  $j_1, \dots, j_n$  of  $1, \dots, n$  for which  $j_1$  and  $j_2 < \dots < j_n$ . The result now follows from the definition of the permanent. Thus the proof is completed.  $\square$

**Example 3.** Let  $F_i(x), i = 1, 2, \dots, n$  be the exponential distribution function

$$F_i(x) = 1 - e^{-\lambda_i x}, \quad x \geq 0, \quad \lambda_i > 0.$$

Then, using (23) one can show that for  $i = 1, \dots, n$ , the MRL function of a system containing three components has the following form:

$$\Phi_{3:3}(t) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} - \frac{1}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}.$$

According to Definition 3, the MRL of the system with exponential distributed components does not depend on  $t$ . The MRL function of a system containing  $n$  components has the form

$$\Phi_{n:n}(t) = \sum_{i=1}^n (-1)^{i+1} \sum_{1 \leq j_1 < \dots < j_i \leq n} \frac{1}{\sum_{k=1}^i \lambda_{j_k}}. \tag{26}$$

An extension of this result is given in Remark 5 of Navarro and Hernandez [15].

#### 4. Numerical example

This section introduces a numerical example for a 2-out-of-3:  $G$  system with Weibull distributed components. Let us consider an airplane that has three engines. Furthermore, assume that at least two engines are required to function for the aircraft to remain airborne. It is assumed that the times to failure of components  $X_i, i = 1, 2, 3$ , are Weibull distributed random variables with parameters  $(\alpha_i, \lambda_i)$ , respectively. The two-parameter Weibull distribution is one of the most useful probability distributions in reliability. It can

be used to model both increasing, and decreasing failure rates.  $\alpha$  is referred to as the shape parameter. If  $\alpha < 1$ , the MRL function is increasing over time. If  $\alpha > 1$ , the MRL function is decreasing over time. If  $\alpha = 1$ , the MRL function is constant over time, that is it has an exponential distribution.

The time to failure  $X_i$  of an engine is said to be Weibull distributed with parameters  $\alpha_i > 0$  and  $\lambda_i > 0$  for  $i = 1, 2, 3$  if the distribution function is given by  $F_i(x) = 1 - e^{-(\lambda_i x)^{\alpha_i}}$ ,  $x \geq 0$ . It is assumed that the scale parameter  $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$  and each component works independently from others. The MRL of an engine at age  $t$  is the average remaining life among those engines which have survived until time  $t$ . The MRL function of the 2-out-of-3:  $G$  system under the condition that all components alive at time  $t > 0$  is

$$H_{(3)}^2(t) = \frac{\sum_{i=0}^1 \binom{3}{i}}{3! \prod_{i=1}^3 e^{-(\lambda t)^{\alpha_i}}} \int_t^{\infty} \text{Per} \left[ \underbrace{e^{-(\lambda t)^{\alpha_i}} - e^{-(\lambda x)^{\alpha_i}}}_i \underbrace{e^{-(\lambda x)^{\alpha_i}}}_{3-i} \right] dx.$$

To study the effect of increasing system level and various parameters on the MRL function of the system, a particular case with  $n = 3$  and  $k = 1, 2, 3$  is analyzed numerically. The MRL function of the  $k$ -out-of-3:  $G$  configuration is calculated versus different parameters of required units. All the computations are made using Maple 5.1. For the components having constant MRL functions, i.e.  $\alpha_1 = \alpha_2 = \alpha_3 = 1$ , the MRL function of the 2-out-of-3:  $G$  system is constant. We observe numerically that, if  $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$  and  $\alpha_i > 1$ , the system has decreasing MRL function, and when  $\alpha_i < 1$  it has increasing MRL function. When either all the components have a linear decreasing MRL function, i.e.  $\alpha_i > 1$ , or two components have linear decreasing MRL function, the system has a linear decreasing MRL function. As the values of  $\alpha_i$  get larger, the values of MRL decrease. Either all components have linear increasing MRL function, i.e.  $\alpha_i < 1$ , or two components have linear increasing MRL function, the system has increasing MRL function.

## 5. Conclusions

In this paper, we derived the MRL function for both parallel and  $k$ -out-of- $n$ :  $G$  structures with nonidentical components. We have focused on exponential, Power and Weibull models which can be used to model monotonic MRL function. The non-monotonic models such as exponentiated Weibull can be the next issue for possible lifetime distribution function of components.

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