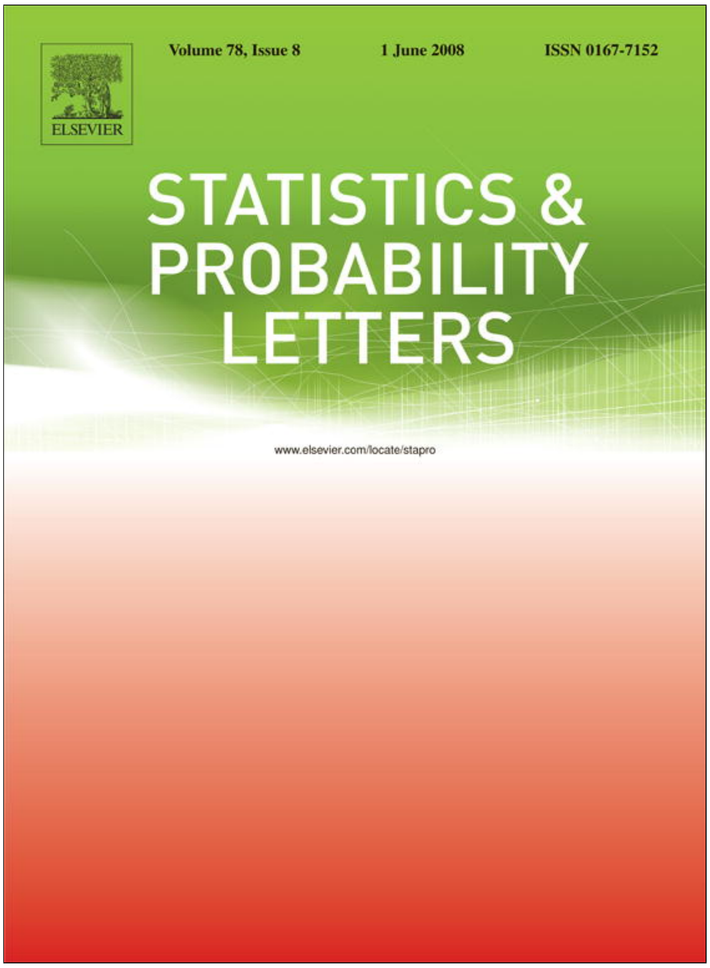
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On the residual lifelengths of the remaining components in an $n-k+1$ out of n system

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Abstract

Suppose that a system consists of n independent components and that the lifelength of the ith component is a random variable X_i $(i = 1, 2, \ldots, n)$. For $k \in \{1, 2, \ldots, n-1\}$, denote by $X_1^{(k)}, X_2^{(k)}, \ldots, X_{n-k}^{(k)}$, the residual lifelengths of the remaining functioning components following the kth failure in the system. We discuss the joint distribution of these exchangeable random variables. In addition, we identify the conditions sufficient to guarantee the independence of the residual lifelengths.

1. Introduction

Consider an $(n-k+1)$ out of n system which will function successfully until k of the components have failed. Consequently, if we denote the lifetimes of the individual components by $X_1, X_2, \ldots, X_n$ then the lifetime of the $(n-k+1)$ out of n system will be represented by the kth order statistic $X_{k:n}$. Detailed discussions of the theory of order statistics may be found in David (1981), Arnold et al. (1992) and David and Nagaraja (2003). For background on the standard theory of reliability, one may refer to the classic text by Barlow and Proschan (1975). Belzunce et al. (1999) and Li and Zuo (2002) give characterizations of nonparametric families of life distributions based on aging and variability orderings of the residual life of a k out of n system.

The classical theory of $n-k+1$ out of n systems assumes that the n lifetimes $X_1, X_2, \ldots, X_n$ of the components of the system are independent and identically distributed (i.i.d.) with common absolutely continuous distribution function F and corresponding density f. With this setup, the time of the first failure will be the first order statistic $X_{1:n}$ and the subsequent times between failures can be identified with the spacings $X_{i:n} - X_{i-1:n}$, $i = 2, 3, \ldots, n$. We will later entertain the possibility that the X_i's are neither independent nor identically distributed, but as is to be expected, the i.i.d. assumption is often crucial for obtaining relatively simple distributional results.

After an $n-k+1$ out of n system fails (i.e. after the kth failure has been observed), it is often reasonable to break down the system and rescue the unfailed components for possible future use in other systems. For example, if we knew

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that an exponential distribution provided a reasonable lifetime model then these used but still functioning components would be just as good as new. We would reuse the used components with even more enthusiasm if we believed that we were in the happy situation in which new is worse than used. In any case, it will be of interest to determine the joint distribution of the residual lifetimes of these unfailed components in order to assess the desirability of reusing them in other systems. Note that even under the classical assumption that the original lifetimes were i.i.d., it will turn out that the residual lifetimes of the unfailed items will be exchangeable, but typically not independent. They will be conditionally independent given the time of the kth failure, but we are not assuming that the time of that failure is known, or equivalently we do not know the time at which the system was switched on, we just know it has stopped functioning because k failures have occurred. Note that if we put the rescued components into a new system, we will need to consider systems with dependent identically distributed component lifetimes, thus justifying a concern with at least this variation on the classical setup of $n-k+1$ out of n systems.

For any $k \in \{1, 2, \dots, n\}$ we will use the notation $X_1^{(k)}, X_2^{(k)}, \dots, X_{n-k}^{(k)}$ to denote the residual lifetimes of the $n-k$ components still functioning at the time of the kth failure. For each k, we may define

$$X_{1:n-k}^{(k)} = \min\{X_1^{(k)}, X_2^{(k)}, \dots, X_{n-k}^{(k)}\}.$$

Upon reflection, it is evident that these $X_{1:n-k}^{(k)}$'s simply represent an alternative description of the spacings of the order statistics of the original sample $X_1, X_2, \dots, X_n$. Thus

$$X_{k+1:n} - X_{k:n} = X_{1:n-k}^{(k)}$$

and

$$X_{k-1:n} = X_{1:n} + X_{1:n-1}^{(1)} + X_{1:n-2}^{(2)} + \cdots + X_{1:n-k}^{(k)}.$$

In the modelling of failure times for components of the system with i.i.d. components, we assume that the failure of one component does not affect the functioning of the remaining ones. If this is not true, for example if the failure of a component puts added stress or load on the remaining components, then models involving so-called sequential order statistics (Kamps, 1995) may be appropriately used in the analysis of the system. We will not consider this variation in the present paper. We begin then with a discussion of residual lifelengths in the classical setup.

2. Main results

2.1. The joint distribution of the residual lifelengths of the remaining components

We begin with $X_1, X_2, \dots, X_n$ i.i.d. with common absolutely continuous distribution F and density f. If we are given $X_{k:n} = x$, then the conditional distribution of the subsequent order statistics $X_{k+1:n}, \dots, X_{n:n}$ is the same as the distribution of order statistics of a sample of size $n-k$ from the distribution F truncated below at x. If we denote by $Y_i^{(k)}$, $i = 1, 2, \dots, n-k$ the randomly ordered values of $X_{k+1:n}, \dots, X_{n:n}$, then given $X_{k:n} = x$, these $Y_i^{(k)}$'s will be i.i.d. with common survival function $\bar{F}(x+y)/\bar{F}(x)$. The residual lifetimes after k failures, $X_1^{(k)}, \dots, X_{n-k}^{(k)}$, may be represented as

$$X_i^{(k)} = Y_i^{(k)} - X_{k:n}, \quad i = 1, 2, \dots, n-k.$$

Using $F_{k:n}$ to denote the distribution function of $X_{k:n}$, we can obtain the joint survival function of the residual lifelengths as follows

$$\begin{aligned}
\bar{F}_n^{(k)}(x_1, x_2, \dots, x_{n-k}) &= P(X_1^{(k)} > x_1, X_2^{(k)} > x_2, \dots, X_{n-k}^{(k)} > x_{n-k}) \\
&= \int_0^\infty P(X_1^{(k)} > x_1, \dots, X_{n-k}^{(k)} > x_{n-k} | X_{k:n} = t) \mathrm{d}F_{k:n}(t) \\
&= \int_0^\infty P(Y_1^{(k)} > x_1 + t, \dots, Y_{n-k}^{(k)} > x_{n-k} + t | X_{k:n} = t) \mathrm{d}F_{k:n}(t) \\
&= \int_0^\infty \left[\prod_{j=1}^{n-k} \frac{\bar{F}(x_j + t)}{\bar{F}(t)}\right] \mathrm{d}F_{k:n}(t). \qquad (1)
\end{aligned}$$

Here and henceforth, the subscript n on a distribution, density or survival function of residual lifetimes denotes the original sample size while the superscript denotes the number of failures that have occurred. From (1) the joint density of the residual lifelengths can be obtained by justifiably differentiating under the integral sign, to get

$$f_n^{(k)}(x_1, x_2, \ldots, x_{n-k}) = \int_0^\infty \left[\prod_{j=1}^{n-k} \frac{f(x_j+t)}{\bar{F}(t)}\right] \mathrm{d}F_{k:n}(t)$$
$$= k\binom{k}{n}\int_0^\infty \prod_{j=1}^{n-k} f(x_j+t)F^{k-1}(t)\mathrm{d}F(t). \tag{2}$$

It is obvious from (1) or (2) that $X_i^{(k)}$'s are exchangeable (it was already obvious since they were conditionally independent given $X_{k:n}$). The common marginal distribution function of the $X_i^{(k)}$'s is

$$F_n^{(k)}(x) = P(X_i^{(k)} \le x)$$
$$= k\binom{k}{n}\int_0^\infty [\bar{F}(t)]^{n-k-1}[F(t+x)-F(t)]F^{k-1}(t)\mathrm{d}F(t). \tag{3}$$

The marginal density of the $X_i^{(k)}$'s can be expressed as

$$f_n^{(k)}(x) = \int_0^\infty \frac{f(t+x)}{\bar{F}(t)} f_{k:n}(t)\mathrm{d}t, \tag{4}$$

where $f_{k:n}$ denotes the density of the kth order statistic $X_{k:n}$.

Under the assumption that the component lifetime distribution is an exponential distribution,it may be observed that this joint density of residual lives has two remarkable features. First, the residual lifetimes are independent. Second, the residual life distribution of a component is the same as the original life distribution of a component. We next investigate the possibilities of characterizing the exponential distribution using these properties.

2.2. *Characterizations*

Assuming that the component lifetime distribution F was an exponential distribution, the residual lifetimes following the kth failure will be independent and will have the same marginal distribution as that of the original lifetimes. It is thus reasonable to ask whether the conditions

(A) $X_1^{(k)} \stackrel{d}{=} X_1$

and

(B) $X_1^{(k)}$ and $X_2^{(k)}$ are independent

are together or separately sufficient to guarantee that the original component lifetime distribution must be exponential. Condition (A) is readily dealt with.

Theorem 1. *If* $X_1^{(k)} \stackrel{d}{=} X_1$ *then* $X_1 \sim$ *exponential* (λ) *for some* $\lambda > 0$.

Proof. If $X_1^{(k)} \stackrel{d}{=} X_1$ then for every $x > 0$,

$$\bar{F}(x) = P(X_1 > x) = P(X_1^{(k)} > x) = \int_0^\infty \frac{\bar{F}(x+t)}{\bar{F}(t)}\mathrm{d}F_{k:n}(t).$$

Thus

$$\int_0^\infty \frac{\bar{F}(x+t)-\bar{F}(x)\bar{F}(t)}{\bar{F}(t)}\mathrm{d}F_{k:n}(t) = 0 \quad \forall x > 0.$$

But this is an integrated Cauchy functional equation (see e.g. Rao and Shanbhag (1994)) and the only solution is of the form $\bar{F}(x) = \mathrm{e}^{-\lambda x}$, $x > 0$ for some $\lambda > 0$. ■

We are able to characterize the exponential distribution using condition (B) by imposing a rather strong regularity condition. Whether this regularity condition can be dispensed with remains an open problem.

Theorem 2. *If* $X_1^{(k)}$ *and* $X_2^{(k)}$ *are independent and if*
(i) $\bar{F}(x)$ *is strictly decreasing on* $(0, \infty)$
and
(ii) *for each* $x > 0$, $\frac{\bar{F}(x+t)}{\bar{F}(t)}$ *is a monotone function of* t, *then* $X_1 \sim$ *exponential* (λ) *for some* $\lambda > 0$.

[Note that a sufficient condition for monotonicity of $\frac{\bar{F}(x+t)}{\bar{F}(t)}$ for every x, is that F has a monotone failure rate, i.e. it is either IFR or DFR.]

Proof. For any $x_1, x_2 > 0$ we have

$$\bar{F}_{(k)}(x_1, x_2) = \int_0^\infty \left[\prod_{j=1}^{2} \frac{\bar{F}(x_j + t)}{\bar{F}(t)}\right] \mathrm{d}F_{k:n}(t)$$

and

$$\bar{F}_{(k)}(x_1) = \int_0^\infty \left(\frac{\bar{F}(x_1 + t)}{\bar{F}(t)}\right) \mathrm{d}F_{k:n}(t),$$

where $\bar{F}_{(k)}(x_1, x_2)$ is the joint distribution function of $X_1^{(k)}$ and $X_2^{(k)}$, $\bar{F}_{(k)}(x_1)$ is the distribution function of $X_1^{(k)}$. Thus if $X_1^{(k)}$ and $X_2^{(k)}$ are independent we have

$$\int_0^\infty \left[\prod_{j=1}^{2} \frac{F(x_j + t)}{\bar{F}(t)}\right] \mathrm{d}F_{k:n}(t) = \int_0^\infty \frac{\bar{F}(x_1 + t)}{\bar{F}(t)} \mathrm{d}F_{k:n}(t) \int_0^\infty \frac{\bar{F}(x_2 + t)}{\bar{F}(t)} \mathrm{d}F_{k:n}(t).$$

We can write this as

$$\operatorname{cov}\left(\frac{\bar{F}(x_1 + X_{k:n})}{\bar{F}(X_{k:n})}, \frac{\bar{F}(x_2 + X_{k:n})}{\bar{F}(X_{k:n})}\right) = 0. \tag{5}$$

Recall what is sometimes called Tchebychev's second inequality. It states that for any random variable X and any two non-decreasing functions ϕ_1 and ϕ_2, then, provided appropriate expectations exist, we have $\operatorname{cov}(\phi_1(X), \phi_2(X)) \geq 0$ with equality if and only if at least one of the random variables $\phi_1(X)$ and $\phi_2(X)$ is degenerate. The same conclusion holds if both ϕ_1 and ϕ_2 are non-increasing.

The assumed monotonicity of $\frac{\bar{F}(x+t)}{\bar{F}(t)}$ for each x, together with Eq. (5) and Tchebychev's second inequality permits us to conclude that for any pair x_1, x_2, at least one of the random variables $\frac{\bar{F}(x_1+X_{k:n})}{\bar{F}(X_{k:n})}$ and $\frac{\bar{F}(x_2+X_{k:n})}{\bar{F}(X_{k:n})}$ is degenerate.

If for every pair x_1, x_2, both of the random variables $\frac{\bar{F}(x_1+X_{k:n})}{\bar{F}(X_{k:n})}$, $\frac{\bar{F}(x_2+X_{k:n})}{\bar{F}(X_{k:n})}$ are degenerate, then it follows that for every x, $\frac{\bar{F}(x+X_{k:n})}{\bar{F}(X_{k:n})}$ is degenerate, say equal to $c(x)$.

If there exists a pair x_1, x_2 for which one of the random variables $\frac{\bar{F}(x_1+X_{k:n})}{\bar{F}(X_{k:n})}$, $\frac{\bar{F}(x_2+X_{k:n})}{\bar{F}(X_{k:n})}$ is not degenerate, then without loss of generality we can assume that $\frac{\bar{F}(x_1+X_{k:n})}{\bar{F}(X_{k:n})}$ is not degenerate, but then for every $x \neq x_1$, we must have $\frac{\bar{F}(x+X_{k:n})}{\bar{F}(X_{k:n})}$ degenerate and equal to $c(x)$, say. Thus for any $y > 0$ and any $x > 0, x \neq x_1$, we have, since $\bar{F}(x)$ is assumed to be decreasing,

$$\frac{\bar{F}(x + y)}{\bar{F}(y)} = c(x). \tag{6}$$

Using the right continuity of $\bar{F}$ we can define $c(x_1) = \lim_{x \downarrow x_1} c(x)$ and conclude that (6) holds for every $x, y > 0$.

But this is Pexider's equation and thus $\bar{F}(x) = k_1 \mathrm{e}^{-\lambda x}$ and $c(x) = k_2 \mathrm{e}^{-\lambda x}$. Finally by considering the limits as $x \to 0$, we conclude that $k_1 = k_2 = 1$. ■

2.3. An extension to exchangeability

Instead of assuming that the X_i's are i.i.d. we may wish to entertain the possibility that they are exchangeable. Indeed some Bayesians might argue that this is almost always more appropriate than an i.i.d. assumption. Provided we interpret the concept of exchangeability in the strict deFinetti sense, then we are really dealing with conditionally independent variables. Thus we assume the existence of a random variable Z with distribution function $G(z)$, such that, given $Z = z$, the X_i's are conditionally i.i.d. with common marginal conditional distribution denoted by $F_z(x)$. Thus the joint distribution of $X_1, X_2, \ldots, X_n$ assumes the form:

$$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n} F_z(x_j) \right] \mathrm{d}G(z). \tag{7}$$

It then follows that the joint survival function of the residual lives of the remaining components after k failures will be of the form

$$\bar{F}_{X_1^{(k)},X_2^{(k)},\ldots,X_{n-k}^{(k)}}(x_1, x_2, \ldots, x_{n-k}) = \int_{-\infty}^{\infty} \left(\int_0^{\infty} \left[\prod_{j=1}^{n-k} \frac{\bar{F}_z(x_j + t)}{\bar{F}_z(t)} \right] \mathrm{d}F_{z;k:n}(t) \right) \mathrm{d}G(z). \tag{8}$$

For most choices of conditional distributions F_z, this expression will be difficult to evaluate. In certain favorable cases analytic results are obtainable.

Example 3. Suppose that, given $Z = z$, the X_i's are conditionally independent exponential (δz) random variables. In addition suppose that $Z \sim \Gamma(\alpha, \lambda)$, i.e.

$$f_Z(z) = \frac{\lambda^{\alpha} z^{\alpha-1} \mathrm{e}^{-\lambda z}}{\Gamma(\alpha)} I(z > 0). \tag{9}$$

In this case we will have

$$\bar{F}^{(k)}_{X_1^{(k)},X_2^{(k)},\ldots,X_{n-k}^{(k)}|Z}(x_1, x_2, \ldots, x_{n-k}|z) = \prod_{j=1}^{n-k} \mathrm{e}^{-\delta z x_i} = \exp\left(-\delta z \sum_{j=1}^{n-k} x_j \right).$$

Consequently the joint density of the residual lives after k failures will be given by

$$\begin{aligned} F^{(k)}_{X_1^{(k)},X_2^{(k)},\ldots,X_{n-k}^{(k)}}(x_1, x_2, \ldots, x_{n-k}) &= \int_0^{\infty} \exp\left(-\delta z \sum_{j=1}^{n-k} x_j \right) \frac{\lambda^k z^{\alpha-1} \mathrm{e}^{-\lambda z}}{\Gamma(\alpha)} \mathrm{d}z \\ &= \left(1 + \frac{\delta}{\lambda} \sum_{j=1}^{n-k} x_j \right)^{-\alpha}, \end{aligned} \tag{10}$$

which is a multivariate Pareto distribution. In (10), the $X_i^{(k)}$'s are identically distributed but only conditionally independent.

2.4. A link with mean residual life functions

For a component X_i with lifetime distribution F, the corresponding mean residual life (MRL) function ψ_F is defined as follows

$$\psi_F(t) = E(X - t|X > t) = \frac{1}{\bar{F}(t)} \int_0^{\infty} x f(t + x) \mathrm{d}x. \tag{11}$$

The MRL function is of much utility in actuarial, survival and reliability settings. For a detailed discussion of the MLR function see Meilijson (1972), Hall and Wellner (1981) and Oakes and Dasu (1990). The MRL function is related to other well-known functions such as the Lorenz curve and the hazard function (cf., (Arnold, 1983)). Recently papers have appeared investigating the mean residual life functions of k out of n systems. See for example, Bairamov et al. (2002), Asadi and Bairamov (2005), Asadi and Bairamov (2006) and Li and Zhao (2006).

In fact the expected value of a residual lifetime after k failures $(X_1^{(k)})$ is directly related to the MRL function of the component lifetime distribution F, i.e. to ψ_F. We have

Theorem 4. $E(X_1^{(k)}) = E(\psi_F(X_{k:n})), k = 1, 2, \ldots, n-1.$

Proof.

$$\begin{aligned} E(X_1^{(k)}) &= \int_0^\infty x f_n^{(k)}(x)\mathrm{d}x \\ &= \int_0^\infty \int_0^\infty x \frac{f(t+x)}{\bar{F}(t)} f_{k:n}(t)\mathrm{d}t\mathrm{d}x \\ &= \int_0^\infty \psi_F(t) f_{k:n}(t)\mathrm{d}t = E(\psi_F(X_{k:n})). \quad \blacksquare \end{aligned}$$

3. Remarks on wearout and reuse of unfailed components

Typically components degrade with usage.

Definition 5. F is said to be new better than used (NBU) if for every $t, x \geq 0$ we have $\bar{F}(x+t) \leq \bar{F}(x)\bar{F}(t)$. If for every $t, x \geq 0$, we have $\bar{F}(x+t) \geq \bar{F}(x)\bar{F}(t)$ then F is said to be new worse than used (NWU).

We use the symbol $\leq_{st}$ to denote stochastic ordering, thus we write $X \leq_{st} Y$ (X is stochastically smaller than Y) if $P(X > x) \leq P(Y > x), \forall x \in \Re$.

Common sense tells us that if components wear out (i.e. if F is NBU) then the residual lifetimes after k failures will be stochastically smaller than the original lifetimes. We may confirm this as follows.

Proposition 6. *If F is NBU(NWU) then $X_1^{(k)} \leq_{st} X_1 (X_1^{(k)} \geq_{st} X_1)$.*

Proof. Assume that F is NBU. We can write the joint distribution function of the survival times as follows

$$F_n^{(k)}(x_1, x_2, \ldots, x_{n-k}) = \int_0^\infty \left[\prod_{j=1}^{n-k} \frac{F(x_j+t) - F(t)}{1 - F(t)} \right] \mathrm{d}F_{k:n}(t).$$

The marginal distribution function of X_1 is obtained by taking the limit as $x_i \to \infty, i = 2, \ldots, n-k$. Thus

$$\begin{aligned} F_n^{(k)}(x_1) &= \int_0^\infty \frac{F(x_1+t) - F(t)}{1 - F(t)} \mathrm{d}F_{k:n}(t) \\ &= \int_0^\infty \frac{\bar{F}(t) - \bar{F}(x_1+t)}{\bar{F}(t)} \mathrm{d}F_{k:n}(t). \end{aligned}$$

Since F is NBU we have $\bar{F}(x_1+t) \leq \bar{F}(x_1)\bar{F}(t)$ and so

$$\begin{aligned} F_n^{(k)}(x_1) &\geq \int_0^\infty \frac{\bar{F}(t) - \bar{F}(x_1)\bar{F}(t)}{\bar{F}(t)} \mathrm{d}F_{k:n}(t) \\ &= [1 - \bar{F}(x_1)] \int_0^\infty \mathrm{d}F_{k:n}(t) \\ &= F(x_1). \quad \blacksquare \end{aligned}$$

Of course if F is both NBU and NWU (i.e. if F is an exponential distribution function) then $X_1^{(k)} \stackrel{d}{=} X_1$.

Suppose that we have on hand $n-k$ unfailed units from an $n-k+1$ out of n system and that we use them to construct an $n-k-k'+1$ out of $n-k$ system. What can we say about the residual lifetimes of the $n-k-k'$ unfailed units from this $n-k-k'+1$ out of $n-k$ system. The joint distribution of the component lifetimes of the

$n-k$ units used to build the second system will be only conditionally independent given $X_{k:n}$ (the failure time of the original $n-k+1$ out of n system). For the second system, built with these used components, the joint distribution of the component lifetimes is a mixture as in (7) with mixing distribution $G(z) = F_{k:n}(z)$ and conditional survival functions $\bar{F}_z(x_j) = \frac{\bar{F}(x_j+z)}{\bar{F}(z)}$. We may then, using (8), obtain the joint survival function of the residual lifetimes of the unfailed items from the second system, i.e. the $n-k-k'+1$ out of $n-k$ system. Thus we obtain

$$\bar{F}_{X_1^{(n-k)}X_2^{(n-k)},\ldots,X_{n-k-k'}^{(n-k)}}(x_1, x_2, \ldots, x_{n-k-k'}) = \int_0^\infty \int_0^\infty \prod_{j=1}^{n-k-k'} \frac{\bar{F}(x_j+t+z)}{\bar{F}(t+z)} \mathrm{d}F_{z;k':n-k}(t)\mathrm{d}F_{k:n}, \tag{12}$$

where $F_{z_j k':n-k}(t)$ denotes the distribution of the k'th order statistic from a sample of size $n-k$ from the distribution with survival function $\bar{F}(x+z)/\bar{F}(z)$. Eventually this simplifies to yield

$$\bar{F}_{X_1^{(n-k)}X_2^{(n-k)},\ldots,X_{n-k-k'}^{(n-k)}}(x_1, x_2, \ldots, x_{n-k-k'}) = \int_0^\infty \prod_{j=1}^{n-k-k'} \frac{\bar{F}(x_j+u)}{\bar{F}(u)} \mathrm{d}F_{k+k':n}(u)$$

confirming the retrospectively obvious result that the residual lives of the remaining components after serving in both systems, correspond in distribution to the residual lives of the remaining components when an $n-k-k'+1$ out of n system has failed. We are indeed waiting first for k failures and then k' more failures among the original n components.

If only $n-p$ of the surviving components from the $n-k+1$ out of n system are used to construct an $n-p-k'+1$ out of $n-p$ system, Eq. (12) must be slightly modified to describe the residual lives of the surviving components in the second system. We will have

$$\bar{F}_{X_1^{(n-p)},X_2^{(n-p)},\ldots,X_{n-p-k'}^{(n-p)}}(x_1, x_2, \ldots, x_{n-p-k'}) = \int_0^\infty \int_0^\infty \prod_{j=1}^{n-p-k'} \frac{\bar{F}(x_j+t+z)}{\bar{F}(t+z)} \mathrm{d}F_{z_j k'+n-p}(t)\mathrm{d}F_{k,n}(z). \tag{13}$$

Only in very special cases will it be possible to simplify this expression. For example, if the original components had exponential (λ) lifetime distributions then the lack of memory property guarantees that the residual lifetimes of the surviving components in the second system (the $n-p-k'+1$ out of $n-p$ system) will again have independent exponential (λ) distributions. Substitution of $\bar{F}(x) = \mathrm{e}^{-\lambda x}$ in (13) will confirm this conclusion.

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