A RESIDUAL LIFE FUNCTION OF A SYSTEM HAVING PARALLEL OR SERIES STRUCTURE

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Abstract. We define the survival and mean residual life function of system consisting of n identical and independent components having series or parallel structure. Let X_i, $i = 1, 2, ..., n$ be the survival time of i th component, such that $X_1, X_2, ..., X_n$ are independent, identically distributed random variables with continuous distribution function F. Let $X_{i:n}$, $i = 1, 2, ..., n$ be the i th smallest among $X_1, X_2, ..., X_n$. The mean residual life function of system having parallel structure function is defined as $\psi_n(t) = E(X_{n:n} - t \mid X_{1:n} > t)$, which can be interpreted as the conditional expectation of residual life length of the system given $X_{1:n} > t$ — none of the components of the system fails at time t. The inverse formula is obtained; i.e.. it is shown that knowledge of ψ_n and ψ_{n-1} for some n, amounts to knowing F. Similar residual life function is defined for a system with functioning series structure, and the inverse formula is given. For parallel structure it is also considered regressing $X_{n:n}$ on $X_{1:n}$ wich can be interpreted as the best predictor of the life length of the system knowing the time when weakest component fails. Some extensions of obtained results to a systems having more complex structure are discussed.

Key Words: Mean residual life function, order statistics, survival function, reliability.

1. Introduction

Consider a technical system A consisting of n identical and mutually independent (from the point of view of failure probabilities) components. Let X_i, $i = 1, 2, ..., n$ be the time up to the failure of i th component, such that $X_1, X_2, ..., X_n$ are independent, identically distributed (i.i.d.) random variables (r.v.) with continuous distribution function (d.f.) F and probability density function f. Let $X_{i:n}$, $i = 1, 2, ..., n$, be the i th smallest among $X_1, X_2, ..., X_n$, so that $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$. The reliability (or survival probability) of i th component corresponding to a mission of duration x, is $\bar{F}(x) \equiv 1 - F(x)$. The corresponding conditional reliability of an i th component at age t is $\bar{F}(x \mid t) \equiv \frac{\bar{F}(x+t)}{\bar{F}(t)}$ if $\bar{F}(t) > 0$. (sf. Barlow and Proshan (1975), p.53). The mean residual life (MRL) function Ψ_F of an component, with life distribution function F pertaining to a life length X, is defined by the following conditional expectation of $X - t$ given $X > t$:

$$\Psi_F(t) = E(X - t \mid X > t),\ t < \omega(F) \equiv \sup\{u : F(u) < 1\}.$$

By definition $\Psi_F(t)$ is the expected remaining life given survival at age t. The MRL function is useful in actuarial analisys, survivorship analisys and reliability. For details sf. Meilijson (1972), Hall and Wellner (1981), Oakes and Desu (1990). The MRL function is related to other well known functions such as the Lorenz curve and the hazard function (sf., Arnold (1983)). Useful identities is known (sf. Cox, 1962 p.128)

$$\bar{F}(x) = \frac{\Psi_F(0)}{\Psi_F(x)} \exp\left\{ -\int_0^x \frac{dt}{\Psi_F(t)} \right\} \tag{1}$$

(sf., also, Arnold and Huang ,(1995); Hinkley, Reid and Snell, (1991)) .

Hall and Wellner (1984) considered the class of distributions with linear mean residual life function,

$$\Psi_F(x) = Ax + b,\ \ (\ A > -1, B > 0).$$

When $A > 0$, $A = 0$ and $-1 < A < 0$, $F(x)$ is a Pareto, an exponential and rescalled beta distribution, respectively. Oakes and Desu (1990) gives two characterizations of a family of survival distributions Hall and Wellner's class in terms of the residual life distributions. Dress and Reiss (1996) show that the empirical MRL function is an inaccurate estimator of the Pareto MRL function if the shape parameter is close to 1 and they investigated the parameters of this conditional distribution such as the median and certain trimmed means. Ebrahimi (1998) introduced a finite population version of the MRL function and hazard function and study Bayesian estimation of these functions.

In this paper, we define the reliability and mean residual life functions of systems having more than one components. First in section 2 and 3 we consider the systems having simple structure i.e. , the systems consisting of n

identical and independent components having series and parallel structure. In section 2 we define the conditional reliability of system having non failure components at time t and corresponding mean residual life function of system, i.e. the conditional expectation of residual life length of the system given "non of the components of the system fails at time t." In section 3 we consider regressing of life length of system on life length of the first failured component, i.e. the best predictor for the life length of a system knowing the time when weakest component fails. For the parallel system the relation between the mean residual life function and the regression function is given. It is shown that knowing mean residual life function for some n and $n-1$ amounts to knowing F, the life distribution of components.

In section 4 we discuss on the extension of our result to a systems having more complex structure, for example k out of n structure , relay structure e.g. . The relation between the structure function and the life length of the system is given. Using this relation one can define the reliability and mean residual life function of a system consisting of n components with given structure function.

Note that proceeding from realword problem the mean residual life function of the system may be defined by another way, for example as the conditional expectation of the residual life length of the system of age t.

We use some basic notations and formulae of theory of order statistics. For more detail, one can refer to, say, David (1981), Arnold, Balakrishnan, Nagaraja (1992).

2. Survival and mean residual life function of systems

2.1. A series structure. Assume that system A has series structure; that is the system functions only each component functions. The survival probability of such a system A corresponding to a mission of duration x is

$$\bar{S}(x) = P\left\{X_{1:n} > x\right\} = \bar{F}^n(x),$$

and the life function is given by

$$S(x) \equiv 1 - \bar{S}(x) = 1 - \bar{F}^n(x).$$

The corresponding conditional reliability of system having non failure element at time t is

$$\bar{S}(x \mid t) = P\left\{X_{1:n} > t + x \mid X_{1:n} > t\right\} = \left[\frac{\bar{F}(t+x)}{\bar{F}(t)}\right]^n, \quad \text{if } \bar{F}(t) > 0. \tag{2}$$

The following result will be useful for further discussion:

Proposition 1. Let $F(x)$ be the exponential distribution function, $F(x) = 1 - \exp(-\lambda x), x \geq 0, \lambda > 0$. Then by the lack of memory property $\bar{F}(t+x) = \bar{F}(t)\bar{F}(x)$ and using (2) one can obtain

$$\bar{S}(x \mid t) = S(x). \tag{3}$$

It is easy to see that exponential distribution is the only distribution with property (3).

Denote $S(x \mid t) = 1 - \bar{S}(x \mid t)$. The mean residual life function of a system A with series structure is defined by the conditional expectation of residual life length

$$\varphi_n(t) = E(X_{1:n} - t \mid X_{1:n} > t),$$

given $X_{1:n} > t$ (all components of A functioning at time t).

Similar identity is obtained from (1) :

$$\bar{S}(x) = \bar{F}^n(x) = \frac{\varphi_n(0)}{\varphi_n(t)} \exp\left\{-\int_0^x \frac{dt}{\varphi_n(t)}\right\} \text{ and}$$

$$\bar{F}(x) = \left[\frac{\varphi_n(0)}{\varphi_n(x)} \exp\left\{-\int_0^x \frac{dt}{\varphi_n(t)}\right\}\right]^{\frac{1}{n}}$$

that is, $\varphi_n(t)$ defines F for some n.

2.2. A parallel structure. Assume that the system A has now a parallel structure; that is, the system goes out of service when all of its components fails— the system functions if only at least one component functions. The survival probability of system corresponding to a mission of duration x is

$$\bar{S}(x) = P\left\{X_{n:n} > x\right\},$$

and the life function is $S(x) = 1 - \bar{S}(x)$.

The conditional probability of survival of system in the interval $(t, t+x)$, with no failing component at time t (the probability that system having non failure elements at time t function at time $t+x$) is

$$\bar{S}(x \mid t) = P\left\{X_{n:n} > t + x \mid X_{1:n} > t\right\}.$$

The conditional probability of system's failing in the interval $(t, t+x]$, with no failing components at time t is

$$S(x \mid t) = P\left\{X_{n:n} \leq t + x \mid X_{1:n} > t\right\}$$

$$= \frac{1}{\bar{F}^n(t)} P\{X_1 < t+x, X_2 < t+x, ...,$$

$$X_n < t+x, X_1 > t, ..., X_n > t\}$$

$$= \left[\frac{F(t+x) - F(t)}{\bar{F}(t)}\right]^n = \left[1 - \frac{\bar{F}(t+x)}{\bar{F}(t)}\right]^n,$$

i.e. ,

$$S(x \mid t) = \left[1 - \frac{\bar{F}(t+x)}{\bar{F}(t)}\right]^n, \text{if } \bar{F}(t) > 0. \tag{4}$$

Proposition 2. Let $F(x)$ be the exponential distribution function;

$$F(x) = 1 - \exp(-\lambda x), \quad x \geq 0, \quad \lambda > 0.$$

Using the lack of memory property $\bar{F}(t+x) = \bar{F}(x)\bar{F}(t)$ one can obtain from (4)

$$S(x \mid t) = F^n(x) = P\{X_{n:n} \leq x\},$$

$$\bar{S}(x \mid t) = 1 - F^n(x) = P\{X_{n:n} > x\} \tag{5}$$

As it is shown in (5) for the exponential distribution, it is also true that

$$\bar{S}(x \mid t) = P\{X_{n:n} > t+x \mid X_{1:n} > t\}$$

$$= P\{X_{n:n} > x\} = \bar{S}(x). \tag{6}$$

It will not be difficult to observe that the exponential distribution is the only one satisfying (6). In fact, let (6) holds. Then from (4) we have

$$1 - \left[1 - \frac{\bar{F}(t+x)}{\bar{F}(t)}\right]^n = 1 - \bar{F}^n(x)$$

and

$$\bar{F}(t+x) = \bar{F}(t)\bar{F}(x).$$

Therefore F must be the exponential distribution function.

Definition. The conditional expectation of residual life length of the system A having parallel structure

$$\psi_n(t) = E(X_{n:n} - t \mid X_{1:n} > t),$$

given $X_{1:n} > t$ (all elements of A function at time t) is called the mean residual life function of parallel system.

By definition, using (4), we have

$$\psi_n(t) = \int x dS(x \mid t)$$

$$= \frac{n}{\bar{F}^n(t)} \int_0^\infty x \left[F(t+x) - F(t)\right]^{n-1} f(t+x) dx$$

$$= \frac{n}{\bar{F}^n(t)} \left[\int_0^\infty (x+t) \left[F(t+x) - F(t)\right]^{n-1} \times \right.$$

$$\times f(t+x) d(t+x) -$$

$$\left. -t \int_0^\infty \left[F(t+x) - F(t)\right]^{n-1} f(t+x) d(x+t) \right]$$

$$= \frac{n}{\bar{F}^n(t)} \left[\int_t^\infty y \left[F(y) - F(t)\right]^{n-1} f(y) dy - \right.$$

$$\left. -t \int_t^\infty \left[F(y) - F(t)\right]^{n-1} f(y) dy \right]$$

$$= \frac{n}{\bar{F}^n(t)} \int_t^\infty y \left[F(y) - F(t)\right]^{n-1} f(y) dy - t,$$

therefore

$$\psi_n(t) = \frac{n}{\bar{F}^n(t)} \int_t^\infty y \left[F(y) - F(t)\right]^{n-1} f(y) dy - t \qquad (7)$$

Example. Let F be the exponential distribution function: $F(x) = 1 - \exp(-\lambda x), x \geq 0, \lambda > 0$. Then using the well known representation

$$X_{n:n} \stackrel{d}{=} \frac{X_1}{n} + \frac{X_2}{n-1} + ... + X_n,$$

where $\stackrel{d}{=}$ denotes the equality in distribution. It is clear from (6) that for the exponential distribution

$$\psi_n(t) = E(X_{n:n} - t \mid X_{1:n} > t) = E(X_{n:n})$$

$$= \frac{1}{\lambda}(\frac{1}{n} + \frac{1}{n-1} + ... + \frac{1}{2} + 1).$$

Theorem 1. Let $\psi_n(t)$ be the mean residual life function of a system having a parallel structure and consisting of n identical and mutually independent components with continuous life distribution function F. Then the following identity holds

$$\bar{F}(x) = \exp\left\{ -\frac{1}{n} \int_0^x \frac{\psi_n'(t) + 1}{\psi_n(t) - \psi_{n-1}(t)} dt \right\}, \tag{8}$$

where $\psi_{n-1}(t)$ is the mean residual life function of similar system having $(n-1)$ components.

Proof. From **(7)** for system having $(n-1)$ components we have

$$\psi_{n-1}(t) = \frac{(n-1)}{\left[\bar{F}(t)\right]^{n-1}} \int_t^\infty y \left[F(y) - F(t)\right]^{n-2} f(y) dy - t \tag{9}$$

Also from (7) one can write

$$\left[\psi_n(t) + t\right] \left[\bar{F}(t)\right]^n = n \int_t^\infty y \left[F(y) - F(t)\right]^{n-1} f(y) dy \tag{10}$$

Differentiating (10) with respect to t we obtain

$$\begin{aligned} &\left[\psi_n'(t) + 1\right] \left[\bar{F}(t)\right]^n - n \left[\psi_n(t) + t\right] \left[\bar{F}(t)\right]^{n-1} f(t) \\ &= -nf(t)(n-1) \int_t^\infty y \left[F(y) - F(t)\right]^{n-2} f(y) dy \\ &\quad -nyf(y) \left[F(y) - F(t)\right]^{n-1} |_{y=t} . \end{aligned} \tag{11}$$

Using the identity (9) , the equation (11) may be rewritten as follows

$$\begin{aligned} &\left[\psi_n'(t) + 1\right] \left[\bar{F}(t)\right]^n - n \left[\psi_n(t) + t\right] \left[\bar{F}(t)\right]^{n-1} f(t) \\ &= -nf(t) \left[\psi_{n-1}(t) + t\right] \left[\bar{F}(t)\right]^{n-1} . \end{aligned} \tag{12}$$

After some derivation from (12), one can obtain

$$\frac{f(t)}{\bar{F}(t)} = \frac{1}{n} \frac{\psi_n'(t) + 1}{\psi_n(t) - \psi_{n-1}(t)} \text{ or}$$

$$\frac{d}{dt}(\ln \bar{F}(t)) = -\frac{1}{n}\frac{\psi_n'(t)+1}{\psi_n(t)-\psi_{n-1}(t)}. \tag{13}$$

Integrating (13) over $[0,x]$ and using $\bar{F}(0)=1$, we obtain

$$\bar{F}(x) = \exp\left\{-\frac{1}{n}\int_0^x \frac{\psi_n'(t)+1}{\psi_n(t)-\psi_{n-1}(t)}dt\right\}$$

Theorem is thus proved.

Remark. Consider (7). It will be not difficult to observe that for a system having parallel structure the problem of characterizing F through only $\psi_n(t)$, for some n, is equivalent to the problem of uniqueness of solution of equation (7) with respect to F. But there are significant difficulties, because the lower limit of the integral depends on t and t belongs also to the integrand function as parameter. For example, in the simplest case $n=2$ we have the equation

$$\psi(t) \equiv \psi_2(t) = \frac{2}{\bar{F}^2(t)}\int_t^\infty y\left[F(y)-F(t)\right]f(y)dy - t \tag{14}$$

Differentiating (14) with respect to t we have

$$\begin{aligned}\left[\psi'(t)+1\right](\bar{F}(t))^2 - 2f(t)\bar{F}(t)(\psi(t)+t)&\\ = -2f(t)\int_t^\infty yf(y)dy&\end{aligned} \tag{15}$$

Differentiating (15) again with respect to t we have

$$\begin{aligned}\psi''(t)\frac{(\bar{F}(t))^2}{f(t)} - \frac{2F(t)f^2(t)-f'(t)(\bar{F}(t))^2}{f^2(t)}\left[\psi'(t)+1\right]&\\ +2f(t)\left[\psi(t)+t\right] - \bar{F}(t)(\psi'(t)+1) = 2tf(t).&\end{aligned} \tag{16}$$

After some derivation from (16) we have the nonlinear differential equation of the form

$$\begin{aligned}\left(\left[y(\psi'(t)+1)\right]^2\right)' - 6((\psi'(t)+1)^2y+&\\ +4\psi(t)(\psi'(t)+1) = 0,&\end{aligned} \tag{17}$$

where $y \equiv y(t) = \frac{\bar{F}(t)}{f(t)}$ is the unknown function, and $\psi(t)$ is a given function. In the cases of $n>2$, the derivation is more difficult.

The similar difficulties exist in problems of characterizing of distributions via regression of order statistics.

3. Regression of life length

By the property of conditional expectation the best unbiased predictor for $X_{m+k:n}$, given $X_{k:n}$, with respect to the squared-error loss is

$$E(X_{m+k:n} \mid X_{k:n}).$$

Historically Fergusson (1967) is the first to consider the problem of determining all d.f.'s for which the regression being linear, i.e.

$$E(X_{m+k:n} \mid X_{k:n}) = aX_{k:n} + b, \quad \text{a.s.} \tag{18}$$

and gives the complete solution for $m = 1$. Wesolowsky and Ahsanullah (1997) give a solution of problem in the case of absolutely continuous distributions for $m = 2$. Their major contribution is the following: Let (18) holds for $k = 2$. Then, according to their finding, if $a > 1$, the distribution becomes the Pareto distribution, if $a < 1$, the distribution turns out to be the Power distribution and if $a = 1$, the distribution is the exponential distribution.

Recently Blaquez and Rebollo (1997) obtained a solution for regression $X_{k+m:n}$ on $X_{k:n}$, $1 \leq k < k + m \leq n$. Their result is the following.

Theorem. (Blaquez and Rebollo, 1997) Let $D_F = \{x : 0 < F(x) < 1\}$ and X be a r.v. with d.f. F which is k times differentiable in D_F, such that

$$E(X_{k+m:n} \mid X_{k:n}) = \beta X_{k:n} + \alpha.$$

Then, except for location and scale parameters,

$$\begin{gathered} F(x) = 1 - |x|^{\delta}, \text{ for } x \in [-1, 0], \text{ if } 0 < \beta < 1 \\ F(x) = 1 - \exp(-x), \text{ for } x \in [0, \infty], \text{ if } \beta = 1 \\ F(x) = 1 - x^{\delta}, \text{ for } x \in [1, \infty], \text{ if } \beta > 1, \end{gathered}$$

where $\delta = (r - (n - m))^{-1}$ and r is the unique real root greater than $m - 1$ of the polynomial equation

$$P_m(z) = \frac{1}{\beta} P_m(n - k),$$

$P_k(z) = z(z - 1)...(z - k + 1).$

It will not difficult to desire the following which constitutes our major contribution to the subject:

Consider regressing $X_{n:n}$ on $X_{1:n}$

$$E(X_{n:n} \mid X_{1:n}),$$

which can be interpreted as the best predictor for life length of a parallel system consisting of n identical and independent component knowing the time of the first failure— i.e., the time when weakest component fails. Denote

$$E(X_{n:n} \mid X_{1:n} = t) = g_n(t). \tag{19}$$

The joint probability density function of $X_{n:n}$ and $X_{1:n}$ is

$$f_{1n}(x,y) = \begin{cases} (n-1)(F(y) - F(x))^{n-2} \times \\ \quad \times f(x)f(y) \quad if \quad x < y \\ 0 \qquad\qquad\qquad otherwise \end{cases}$$

(sf. David ,1981). It will be routine matter to see from (19) that

$$g_n(t) = \frac{n-1}{(\bar{F}(t))^{n-1}} \int_t^{\infty} y(F(y) - F(t))^{n-2} f(y)dy. \tag{20}$$

It will not difficult to see from (9) and (20) that

$$g_n(t) = \Psi_{n-1}(t) + t. \tag{21}$$

One can obtain a similar identity as in Theorem 1 as follows:

Theorem 2. Under integrability assumptions, it is true that

$$\bar{F}(x) = \exp\left\{-\int_0^x \frac{g_2'(t)}{g_2(t) - t} dt\right\} \quad \text{if} \quad n = 2 \quad \text{and}$$

$$\bar{F}(x) = \exp\left\{-\frac{1}{n-1}\int_0^x \frac{g_n'(t)}{g_n(t) - g_{n-1}(t)} dt\right\} \text{ if } n > 2.$$

Proof. Let $n = 2$. Then from (20) we have

$$g_2(t)\bar{F}(t) = \int_t^{\infty} yf(y)dy. \tag{22}$$

Differentiating (21) with respect to t one can obtain $g_2'(t)\bar{F}(t)-f(t)g_2(t) = -tf(t)$ and

$$\frac{g_2'(t)}{g_2(t)-t} = \frac{f(t)}{\bar{F}(t)}. \tag{23}$$

Integrating (22) over $[0,x]$ we obtain $\bar{F}(x) = \exp\left\{-\int\limits_0^x \frac{g_2'(t)}{g_2(t)-t}dt\right\}$.

Now let $n > 2$. Then from (20) one can write

$$g_{n-1}(t) = \frac{n-2}{(\bar{F}(t))^{n-2}} \int\limits_t^\infty y(F(y)-F(t))^{n-3} f(y)dy \tag{24}$$

and

$$g_n(t)(\bar{F}(t))^{n-1} = (n-1)\int\limits_t^\infty y(F(y)-F(t))^{n-2} f(y)dy. \tag{25}$$

Differentiating (25) with respect to t we have

$$\begin{aligned} &g_n'(t)(\bar{F}(t))^{n-1} - (n-1)g_n(t)(\bar{F}(t))^{n-2}f(t) \\ &= -(n-1)(n-2)\int\limits_t^\infty y(F(y)-F(t))^{n-3}f(y)dy. \end{aligned} \tag{26}$$

Using (24) in (26) we can obtain

$$\begin{aligned} &g_n'(t)(\bar{F}(t))^{n-1} - (n-1)g_n(t)(\bar{F}(t))^{n-2}f(t) \\ &= -(n-1)f(t)(\bar{F}(t))^{n-2}g_{n-1}(t). \end{aligned} \tag{27}$$

From (26) it follows that

$$\frac{f(t)}{\bar{F}(t)} = \frac{1}{n-1}\frac{g_n'(t)}{g_n(t)-g_{n-1}(t)}. \tag{28}$$

Integrating (27) in $[0,x]$ we obtain the assertion of theorem.

4. Discussion on the extension of results to the systems having complex structure

Consider the system consisting of n components Following the notation of Barlow and Proschan (sf. Barlow and Proschan, 1974, p. 1-2) define the following variables: the binary indicator variable

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is failed} \end{cases}, \quad i = 1, 2, ..., n$$

and the binary variable

$$\Phi(x_1, x_2, ..., x_n) = \begin{cases} 1 & \text{if the system functioning} \\ 0 & \text{if the system is failed} \end{cases}.$$

$\Phi(x_1, x_2, ..., x_n)$ is called the structure function of the system. For the series structure the structure function is given by $\Phi(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} x_i = \min(x_1, x_2, ..., x_n)$, $x_1 x_2 = \min(x_1, x_2)$. For parallel structure the structure function has a form $\Phi(x_1, x_2, ..., x_n) = \coprod_{i=1}^{n} x_i = \max(x_1, x_2, ..., x_n)$, where $\coprod_{i=1}^{n} x_i = 1 - \prod_{i=1}^{n}(1 - x_i)$, $x_1 \sqcup x_2 = 1 - (1 - x_1)(1 - x_2) = \max(x_1, x_2)$.

A k out of n structure. A k out of n structure functions if and only if at least k of the n components function. The structure function is given by

$$\Phi(x_1, x_2, ..., x_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{if } \sum_{i=1}^{n} x_i < k \end{cases}$$

or equivalently

$$\Phi(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} x_i \text{ for } k = n,$$

while

$$\begin{gathered} \Phi(x_1, x_2, ..., x_n) = x_1 x_2 ... x_k \sqcup x_1 x_2 ... x_{k-1} x_{k+1} \sqcup ... \\ \sqcup x_{n-k+1} ... x_n = \max\{\min(x_1, x_2, ..., x_k), \\ \min(x_1, x_2, ..., x_{k-1}, x_{k+1}), ..., \min(x_{n-k+1}, ..., x_n)\} \end{gathered} \tag{29}$$

for $1 \leq k \leq n$, where every choice of k out of the n x's appears one exactly. It is clear that a series structure is an $n-$ out of n structure and a parallel structure is 1 out of n structure.

Let X_i be the life length of i th component. It is not difficult to observe that the system having the structure function $\Phi(x_1, x_2, ..., x_n)$ given in terms of "max and min" functions at time t if and only if

$$\Phi(X_1, X_2, ..., X_n) > t.$$

For example the system with structure function given as (29) functions if and only if

$$\max\{\min(X_1, X_2, ..., X_k), \min(X_1, X_2, ..., X_{k-1}, X_{k+1}), \\ ..., \min(X_{n-k+1}, ..., X_n) > t\}.$$

For simplicity let us consider 2 out of 3 structure. The survival probability of such a system corresponding to a mission of duration is

$$\bar{S}(x) = P\{\max(\min(X_1, X_2), \min(X_2, X_3), \\ \min(X_1, X_3)) > x\},$$

the conditional probability of survival of system in the interval $(t, t+x)$, with non failing components at time t is

$$\bar{S}(x \mid t) = P\{\max(\min(X_1, X_2), \min(X_2, X_3), \\ \min(X_1, X_3)) > t + x \mid \min(X_1, X_2, X_3) > t\}.$$

The $S(x \mid t) = 1 - \bar{S}(x \mid t)$ can be interpreted as the conditional probability of system failing in the interval $(t, t+x]$ with non failing component at time t. The mean residual life function of this system can be compute as follows:

$$\Psi(t) = E\{\max(\min(X_1, X_2), \min(X_2, X_3), \\ \min(X_1, X_3)) - t \mid \min(X_1, X_2, X_3) > t\} \\ = \int x dS(x \mid t)$$

Let us consider another example, so called Relay structure:

Relay structure. Let system has a relay structure with structure function $\Phi(x_1, x_2) = x_1(x_2 \sqcup x_3) = \min(x_1, \max(x_2, x_3))$. It is known that relays are subject to two kinds of failure: failure to close and failure to open (sf. Barlow and Proschan, P.12). Let X_i, $i = 1, 2, 3$ be the life length of i th component. Assume that X_i, $i = 1, 2, 3$ are i.i.d. r.v.'s with distribution function F. It is clear that the system functions at time t if and only if

$$\Phi(X_1, X_2) = \min(X_1, \ \max(X_2, X_3)) > t.$$

The conditional probability of survival of system in the interval $(t, t+x)$, with no failing component at time t is

$$
\begin{gathered}
\bar{S}(x \mid t) = P\{X_1 > t + x, \\
\max(X_2, X_3) > t + x \mid X_1 > t,\, X_2 > t, X_3 > t\} \\
= \frac{P\{X_1 > t+x, \max(X_2,X_3) > t+x, X_1 > t, X_2 > t, X_3 > t\}}{P\{X_1 > t, X_2 > t, X_3 > t\}} = \\
= \frac{1}{P\{X_{1:3} > t\}} P\{X_1 > t + x, X_3 > t, X_2 > t + x \\
\cup X_1 > t + x, X_2 > t, X_3 > t + x\} \\
= \frac{1}{P\{X_{1:3} > t\}} (P\{X_1 > t + x, X_3 > t, X_2 > t + x\} \\
+ P\{X_1 > t + x, X_2 > t, X_3 > t + x\} \\
- P\{X_1 > t + x, X_2 > t + x, X_3 > t + x\}) \\
= 2\left(\frac{\bar{F}(t + x)}{\bar{F}(t)}\right)^2 - \left(\frac{\bar{F}(t + x)}{\bar{F}(t)}\right)^3
\end{gathered}
$$

The conditional probability of systems failing in the interval $(t, t + x)$, with no failing component at time t is

$$
\begin{gathered}
S(x \mid t) = 1 - \bar{S}(x \mid t) \\
= 1 - 2\left(\frac{\bar{F}(t+x)}{\bar{F}(t)}\right)^2 + \left(\frac{\bar{F}(t+x)}{\bar{F}(t)}\right)^3 .
\end{gathered} \tag{30}
$$

The mean residual life function will be compute as

$$
\Psi(t) = \int x dS(x \mid t).
$$

It is not difficult to see that for the exponential distribution function $F(x) = 1 - \exp(-\lambda x), x \geq 0$

$$
S(x \mid t) = 1 - 2\exp(-2\lambda x) + \exp(-3\lambda x)
$$

and

$$
\begin{aligned}
\Psi(t) = \int x dS(x \mid t) = 4\lambda \int x e^{-2\lambda x} dx + 3\lambda \int x e^{-3\lambda x} dx \\
= \frac{7}{9\lambda}.
\end{aligned}
$$

It is possible another examples.

5. Concluding remarks

One can note that proceeding from realword problems one can define the MRL function of a system consisting of n components by another way, for example as the conditional expectation of residual life length of system functioning at time t (according to structure function some of components may be failing at time t, but the system functions at this time). Let X_i, $(i = 1, 2, ..., n)$ be the life length of i th component. $\Phi(x_1, x_2, ..., x_n)$ be the structure function given in terms "max" and "min". Then $\Phi(X_1, X_2, ..., X_n)$ express the life length of a system. Then according to the above consideration the survival probability of such a system corresponding to a mission of duration x may be defined as follows:

$$\bar{S}(x) = P\left\{\Phi(X_1, X_2, ..., X_n) > x\right\}$$

$$\bar{S}(x \mid t) = P\left\{\Phi(X_1, X_2, ..., X_n) - t \mid \Phi(X_1, X_2, ..., X_n) > t\right\}$$

For example consider series and parallel structure. It is clear hat for the series structure nothing will change, because $\Phi(x_1, x_2, ..., x_n) = \min(x_1, x_2, ..., x_n)$. For parallel structure the conditional reliability of a system of age t will have a form

$$\bar{S}(x \mid t) = P\left\{X_{n:n} > t + x) \mid X_{n:n} > t\right\};$$

the conditional probability of failure during the next interval of duration x of a system of age t is

$$S(x \mid t) = P\left\{X_{n:n} \le t + x) \mid X_{n:n} > t\right\}.$$

The mean residual life time function can be defined as

$$\Psi(t) = E(X_{n:n} - t \mid X_{n:n} > t) =$$

$$= \int_t^\infty x dS(x \mid t) - t = n \int_t^\infty x(F(x))^{n-1} dF(x) - t.$$

It is not difficult to see that

$$F(x) = (1 - \bar{S}(x))^{\frac{1}{n}}$$
$$= \left(1 - \frac{\Psi_F(0)}{\Psi_F(x)} \exp\left\{-\int_0^x \frac{dt}{\Psi_F(t)}\right\}\right)^{\frac{1}{n}}.$$

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