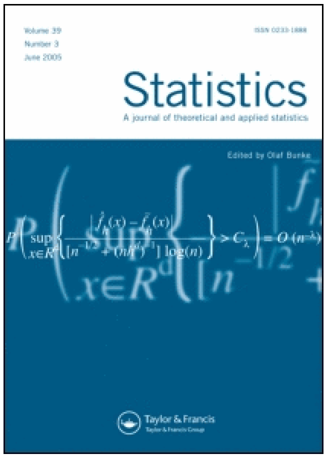

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Distributions of exceedances of generalized order statistics

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Distributions of exceedance statistics based on minimal spacing of generalized order statistics are obtained in a random threshold model. As the special cases of the generalized order statistics the ordinary order statistics, Progressively Type II right censored order statistics and record values are considered. The results obtained in the paper imply previous results on exceedance statistics for the variety of models of ordered random variables.

Keywords: Order statistics, Generalized order statistics, Spacings

1. Introduction

Udo Kamps [1, 2] has introduced generalized order statistics as random variables having certain joint density function which includes as a special case the joint density functions of many models of ordered random variables. Let F be an absolutely continuous distribution function with density function f. The random variables $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are called generalized order statistics based on F, if their joint density function is given by

$$f^{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1, x_2, \ldots, x_n) = k(1 - F(x_n))^{k-1} f(x_n) \prod_{i=1}^{n-1} \gamma_i (1 - F(x_i))^{m_i} f(x_i), \tag{1}$$

on the cone $F^{-1}(0) < x_1 \le x_2 \le \ldots \le x_n < F^{-1}(1)$ of R^n with parameters $n \in N$, $n \ge 2$, $k > 0, \tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in N^{n-1}$, $M_r = \sum_{i=r}^{n-1} m_i$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. Denote $c_{r-1} = \prod_{i=1}^{r} \gamma_i$, $r = 1, 2, \ldots, n-1$ and $\gamma_n = k$. The distribution theory of generalized order statistics are given in Kamps and Cramer [3]. Let $a_j(r) = \prod_{i=1, i \ne j}^{r} (1/(\gamma_i - \gamma_j))$ for $1 \le j \le r \le n$, $\gamma_i \ne \gamma_j$, $a_i^{(r)}(s) = \prod_{j=r+1, i \ne j}^{s} 1/(\gamma_j - \gamma_i)$ for $r + 1 \le i \le s \le n$, and $\prod_{\varnothing} = 1$. The probability density function (pdf) of the rth

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generalized order statistic $X(r, n, \tilde{m}, k)$ is

$$f^{X(r,n,\tilde{m},k)}(x) = c_{r-1} f(x) \sum_{j=1}^{r} a_j(r)(1 - F(x))^{\gamma_j - 1} \text{ for } 1 \le r \le n.$$

The cumulative distribution function (cdf) of $X(r, n, \tilde{m}, k)$ is

$$F^{X(r,n,\tilde{m},k)}(x) = 1 - c_{r-1} \sum_{j=1}^{r} \frac{a_j(r)}{\gamma_j}(1 - F(x))^{\gamma_j}.$$

The joint pdf of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$ is

$$f^{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x_r, x_s) = c_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^{(r)}(s)(1 - F(x_r))^{\gamma_j} \left(\frac{1 - F(x_s)}{1 - F(x_r)}\right)^{\gamma_i} \times \frac{f(x_r)}{1 - F(x_r)} \frac{f(x_s)}{1 - F(x_s)}$$

for $1 \le r < s \le n$ and $x_r \le x_s$.

It is not easy to find a natural interpretation of generalized order statistics in terms of observed random samples but an interesting special case is the Progressive Type II censored order statistics. This model is one of the most applicable general models of ordered random variables and is useful in reliability and life time studies. Denote by $X_{1:n:N}^{(\mathbf{R})}, X_{2:n:N}^{(\mathbf{R})}, \ldots, X_{n:n:N}^{(\mathbf{R})}$ the Progressive Type II right censored order statistics from a sample $X_1, X_2, \ldots, X_N$ with progressive censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_n)$. A nice description of details of the theory, methods and applications of Progressive censoring can be found in Balakrishnan and Aggarwala [4]. The choice of parameters in equation (1) as $m_i = -1$, $i = 1, 2, \ldots, n-1$, $k = 1$ gives a well studied in the literature model of record values.

The details of the theory of records can be found in Galambos [5], Nagaraja [6], Nevzorov [7], Ahsanullah [8], Arnold *et al.* [9] among others.

There are also other special cases of generalized order statistics such as sequential order statistics, kth records, Pfiefer's record model etc.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random sample of size n from an absolutely continuous distribution with a common cdf $F_X(\cdot)$. Denote the ordinary order statistics of $X_1, X_2, \ldots, X_n$ by $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$. Let $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ be another sample of size m from the same distribution independent from X. It is well known that for $1 \le i < j \le n$

$$P\{X_{n+1} \in (X_{i:n}, X_{j:n})\} = \frac{j - i}{n + 1}, \tag{2}$$

i.e. $(X_{i:n}, X_{j:n})$ is a distribution free confidence interval containing the future observation in the class of all absolutely continuous distribution functions $\Im_c$. Let $X_1, X_2, \ldots, X_n, X_{n+1}, \ldots, X_{n+m}$ be the sample from the distribution with continuous cdf F. Denote

$$\xi_l = \begin{cases} 1, & \text{if } \quad X_{i:n} \le X_{n+l} \le X_{j:n} \\ 0, & \quad\text{otherwise,} \end{cases} \qquad l = 1, 2, \ldots, m$$

and

$$S_m = \sum_{l=1}^{m} \xi_l.$$

The distribution of S_m is

$$P\{S_m = k\} = \frac{j-i}{j-i+k} \frac{\binom{m}{k}\binom{n}{j-i}}{\binom{m+n}{k+j-i}}, \qquad k = 0, 1, 2, \ldots, m \tag{3}$$

(see *e.g.*, refs. [10–12]. The asymptotic distribution of S_m is

$$\lim_{m\to\infty} \sup_{0\le x\le 1} \left| P\left\{ \frac{S_m}{m} \le x \right\} - I_x\,(j-i, n-j+i+1) \right| = 0,$$

where

$$I_x(j-i, n-j+i+1) = \frac{1}{B(j-i, n-j+i+1)} \int_0^x t^{j-i-1}(1-t)^{n-j+i}\,\mathrm{d}t,$$

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t.$$

Bairamov [13] considered the exceedance statistics in record model and derived the exact and asymptotic distriutions. Wesolowski and Ahsanullah [14] considered more general models of exceedances based on records of two independent sequences $\{X_n\}_{n\ge1}$ and $\{Y_n\}_{n\ge1}$ with distribution functions F_X and F_Y, respectively. They apply their results for characterizing equidistributions. Bairamov and Eryilmaz [15] consider the exceedance model based on spacing having minimal length. Stepanov [16] considered the multiple exceedance statistics in a record model and derived the joint distributions.

In this paper we consider the generalized order statistics $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ based on the continuous distribution function F and observations $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ from the population having distribution function F. For $m_i = 0$, $i = 1, 2, \ldots, n-1$, $k = 1$ the random variables $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are ordinary order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ of the sample $X_1, X_2, \ldots, X_n$ and we assume that $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ are new observations independent from $X_1, X_2, \ldots, X_n$; for $k = R_n + 1$, $m_i = R_i$, $i = 1, 2, \ldots, m-1$ and $\gamma_r = N - \sum_{i=1}^{r-1} R_i - r + 1$, $r = 2, 3, \ldots, n-1$ and $\gamma_1 = N$ the $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are Progressive Type II censored order statistics based on the sample $X_1, X_2, \ldots, X_n$ with distribution function F and we assume that $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ are new observations from F independent from $X_1, X_2, \ldots, X_n$; for $m_i = -1$, $i = 1, 2, \ldots, n-1$, $k = 1$ the random variables $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are records $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ of the sequence $X_1, X_2, \ldots, X_n, \ldots$ and we assume that $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ are new observations $X_{U(n)+1}, X_{U(n)+2}, \ldots, X_{U(n)+m}$ coming after the rth record etc.

2. Exceedance statistics based on minimal spacing

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with continuous cdf F and pdf f. Let $X(r, n, \tilde{m}, k)$ be the rth generalized order statistic based on f. Consider the spacings $X(1, n, \tilde{m}, k) - X(0, n, \tilde{m}, k)$, $X(2, n, \tilde{m}, k) - X(1, n, \tilde{m}, k), \ldots,$ $X(n, n, \tilde{m}, k) - X(n-1, n, \tilde{m}, k)$ with $X(0, n, \tilde{m}, k) = 0$. Define a random variable υ as the index of a spacing having minimal length.

Ahsanullah [17] has shown that if the underlying distribution is exponential with parameter λ, then $X(j,n,\tilde{m},k) - X(j-1,n,\tilde{m},k)$, $j = 1,2,\ldots,n$, $(X(0,n,\tilde{m},k) = 0)$ are independent. Furthermore, for $1 < r \leq n$, the statistics $X(r,n,\tilde{m},k) - X(r-1,n,\tilde{m},k)$ and $X(r-1,n,\tilde{m},k)$ are also independent.

The following lemma is useful in our work.

LEMMA 1 *Let* $X(j,n,\tilde{m},k) - X(j-1,n,\tilde{m},k) = W_j$, $j = 1,2,\ldots,n$, $(W_1 = X(1,n,\tilde{m},k))$ *and* $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$, $\lambda > 0$. *The cdf of* W_j *is*

$$F^{W_j}(w) = 1 - \exp(-\lambda \gamma_j w),$$

where $w \geq 0, \lambda > 0$.

The following theorems can be easily concluded from this lemma.

THEOREM 1 *Let* $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$, $\lambda > 0$. *Then*

$$P\{v = r\} = \frac{\gamma_r}{\sum_{i=1}^{n} \gamma_i}, \quad r = 1,2,\ldots,n.$$

Let $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$, $\lambda > 0$. *Then the p.d.f of* $X(j-1,n,\tilde{m},k)$ *is*

$$f^{X(j-1,n,\tilde{m},k)}(x) = \lambda c_{j-2} \sum_{i=1}^{j-1} a_i(j-1) e^{-\lambda \gamma_i x}.$$

LEMMA 2 *Let* $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ *be* m *observations from population with cdf* F. *Assume that the generalized order statistics* $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k), \ldots, X(n,n,\tilde{m},k)$ *and* $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ *are independent. Let* $(X(v-1,n,\tilde{m},k), X(v,n,\tilde{m},k))$ *be the interval with minimal length. If* $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$, $\lambda > 0$, *then for* $1 \leq l \leq m$

$$\begin{aligned}
&P\{X_{n+1}, X_{n+2}, \ldots, X_{n+l} \in (X(v-1,n,\tilde{m},k), X(v,n,\tilde{m},k)),\\
&\qquad X_{n+l+1}, X_{n+l+2}, \ldots, X_{n+m} \notin (X(v-1,n,\tilde{m},k), X(v,n,\tilde{m},k))\}\\
&= \gamma_1 B\left(m - l + \sum_{i=1}^{n} \gamma_i, l+1\right) + \sum_{t=0}^{m-l} \binom{m-l}{t} B\left(m - l - t + \sum_{j=1}^{n} \gamma_j, l+1\right)\\
&\quad \times \sum_{r=2}^{n} \sum_{i=1}^{r-1} c_{r-1} a_i(r-1) B(\gamma_i + m - t, t+1),
\end{aligned}$$

where $B(a,b)$ *is the* beta function.

Proof One observes that the probability of the event

$$\begin{aligned}
&\{X_{n+1}, X_{n+2}, \ldots, X_{n+l} \in (X(v-1,n,\tilde{m},k), X(v,n,\tilde{m},k)),\\
&\qquad X_{n+l+1}, X_{n+l+2}, \ldots, X_{n+m} \notin (X(v-1,n,\tilde{m},k), X(v,n,\tilde{m},k))\}
\end{aligned}$$

can be calculated as follows:

$$
\begin{aligned}
&P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \in (X(v-1,n,\tilde{m},k), X(v,n,\tilde{m},k)), X_{n+l+1}, X_{n+l+2}, \dots, X_{n+m} \\
&\quad \notin (X(v-1,n,\tilde{m},k), X(v,n,\tilde{m},k))\} \\
&= P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \in (X(0,n,\tilde{m},k), X(1,n,\tilde{m},k)), X_{n+l+1}, X_{n+l+2}, \dots, X_{n+m} \\
&\quad \notin (X(0,n,\tilde{m},k), X(1,n,\tilde{m},k)), v=1\} + \sum_{r=2}^{n} P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \\
&\quad \in (X(r-1,n,\tilde{m},k), X(r,n,\tilde{m},k)), X_{n+l+1}, X_{n+l+2}, \dots, X_{n+m} \\
&\quad \notin (X(r-1,n,\tilde{m},k), X(r,n,\tilde{m},k)), v=r\}.
\end{aligned}
$$

By using the independence of spacings we can write

$$
\begin{aligned}
&P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \in (X(0,n,\tilde{m},k), \\
&\quad X(1,n,\tilde{m},k)), X_{n+l+1}, X_{n+l+2}, \dots, X_{n+m} \notin (X(0,n,\tilde{m},k), X(1,n,\tilde{m},k)), v=1\} \\
&\quad = P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \in (0, X(1,n,\tilde{m},k)), X_{n+l+1}, X_{n+l+2}, \dots, X_{n+m} \\
&\qquad \notin (0, X(1,n,\tilde{m},k)), X(1,n,\tilde{m},k) \le W_2, X(1,n,\tilde{m},k) \\
&\quad \le W_3, \dots, X(1,n,\tilde{m},k) \le W_n\} \\
&\quad = \int_0^\infty P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \in (0,t_1), X_{n+l+1}, X_{n+l+2}, \dots, X_{n+m} \notin (t_1,\infty), \\
&\qquad \times t_1 \le W_1, t_1 \le W_2, \dots, t_1 \le W_n\} f^{X(1,n,\tilde{m},k)}(t_1)\mathrm{d}t_1 \\
&\quad = \int_0^\infty (1-e^{-\lambda t_1})^l e^{-\lambda t_1(m-l+\sum_{i=2}^n \gamma_i)} f^{X(1,n,\tilde{m},k)}(t_1)\mathrm{d}t_1 \\
&\quad = \int_0^\infty (1-e^{-\lambda t_1})^l e^{-\lambda t_1(m-l+\sum_{i=2}^n \gamma_i)} \lambda\gamma_1 e^{-\lambda t_1}\mathrm{d}t_1 \\
&\quad = \lambda\gamma_1 \int_0^\infty (1-e^{-\lambda t_1})^l e^{-\lambda t_1(m-l+\sum_{i=1}^n \gamma_i)}\mathrm{d}t_1 = \gamma_1 B\left(m-l+\sum_{i=1}^n \gamma_i, l+1\right).
\end{aligned}
$$

Similarly, by using the independence of $X(r,n,\tilde{m},k) - X(r-1,n,\tilde{m},k)$ and $X(r-1,n,\tilde{m},k)$ for $1 < r \le n$ we have

$$
\begin{aligned}
&P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \in (X(r-1,n,\tilde{m},k), X(r,n,\tilde{m},k)), X_{n+l+1}, X_{n+l+2}, \dots, X_{n+m} \\
&\quad \notin (X(r-1,n,\tilde{m},k), X(r,n,\tilde{m},k)), v=r\} \\
&= P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \in (X(r-1,n,\tilde{m},k), X(r,n,\tilde{m},k)), \\
&\qquad X_{n+l+1}, X_{n+l+2}, \dots, X_{n+m} \notin (X(r-1,n,\tilde{m},k), X(r,n,\tilde{m},k)), \\
&\qquad W_r \le W_1, W_r \le W_2, \dots, W_r \le W_n\} \\
&= \int_0^\infty \int_0^\infty P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \in (t_2, t_1+t_2), X_{n+l+1}, X_{n+l+2}, \dots, X_{n+m} \\
&\quad \notin (t_2, t_1+t_2), t_1 \le W_1, t_1 \le W_2, \dots, t_1 \le W_n\} f^{W_r}(t_1) f^{X(r-1,n,\tilde{m},k)}(t_2)\mathrm{d}t_1\mathrm{d}t_2 \\
&= \int_0^\infty \int_0^\infty \sum_{t=0}^{m-l} \binom{m-l}{t} P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \in (t_2, t_1+t_2),
\end{aligned}
$$

exactly t of $X_{n+i} \in (0, t_2)$, exactly $(m-l-t)$ of

$$X_{n+i} \in (t_1+t_2, \infty), t_1 \leq W_1, t_1 \leq W_2, \dots, t_1 \leq W_n\} f^{W_r}(t_1) f^{X(r-1,n,\tilde{m},k)}(t_2)\mathrm{d}t_1\mathrm{d}t_2$$

$$= \sum_{t=0}^{m-l} \binom{m-l}{t} \int_0^\infty \int_0^\infty (F(t_1+t_2) - F(t_2))^l (F(t_2))^t$$

$$\times (1 - F(t_1+t_2))^{m-l-t} \frac{\prod_{j=1}^n (1 - F^{W_j}(t_1))}{1 - F^{W_r}(t_1)} f^{W_r}(t_1) f^{X(r-1,n,\tilde{m},k)}(t_2)\mathrm{d}t_1\mathrm{d}t_2$$

$$= \sum_{t=0}^{m-l} \binom{m-l}{t} \int_0^\infty \int_0^\infty (1 - \mathrm{e}^{-\lambda t_1})^l (1 - \mathrm{e}^{-\lambda t_2})^t \frac{\mathrm{e}^{-\lambda\left(m-l-t+\sum_{j=1}^n \gamma_j\right)t_1}}{\mathrm{e}^{-\lambda\gamma_r t_1}}$$

$$\times \mathrm{e}^{-\lambda(m-t)t_2} \lambda\gamma_r e^{-\lambda\gamma_r t_1} \lambda c_{r-2} \sum_{j=1}^{r-1} a_j(r-1)\mathrm{e}^{-\lambda\gamma_j t_2}\,\mathrm{d}t_1\,\mathrm{d}t_2$$

$$= \lambda^2 c_{r-1} \sum_{t=0}^{m-l} \sum_{j=1}^{r-1} \binom{m-l}{t} a_j(r-1) \left(\int_0^\infty \mathrm{e}^{-\lambda\left(m-l-t+\sum_{j=1}^n \gamma_j\right)t_1} (1 - \mathrm{e}^{-\lambda t_1})^l \mathrm{d}t_1 \right)$$

$$\times \left(\int_0^\infty \mathrm{e}^{-\lambda(\gamma_j+m-t)t_2} (1 - \mathrm{e}^{-\lambda t_2})^t \mathrm{d}t_2 \right)$$

$$= \lambda^2 c_{r-1} \sum_{t=0}^{m-l} \sum_{j=1}^{r-1} \binom{m-l}{t} a_j(r-1) \frac{B\left(m-l-t+\sum_{j=1}^n \gamma_j, l+1\right)}{\lambda}$$

$$\times \frac{B(\gamma_j+m-t, t+1)}{\lambda}$$

$$= c_{r-1} \sum_{t=0}^{m-l} \binom{m-l}{t} B\left(m-l-t+\sum_{j=1}^n \gamma_j, l+1\right) \sum_{j=1}^{r-1} a_j(r-1) B(\gamma_j+m-t, t+1)\cdot$$

Therefore,

$$P\{X_{n+1}, X_{n+2}, \dots, X_{n+l} \in (X(v-1,n,\tilde{m},k), X(v,n,\tilde{m},k)), X_{n+l+1}, X_{n+l+2}, \dots, X_{n+m} \notin (X(v-1,n,\tilde{m},k), X(v,n,\tilde{m},k))\}$$

$$= \gamma_1 B\left(m-l+\sum_{i=1}^n \gamma_i, l+1\right) + \sum_{r=2}^n c_{r-1} \sum_{t=0}^{m-l} \binom{m-l}{t}$$

$$\times B\left(m-l-t+\sum_{j=1}^n \gamma_j, l+1\right) \sum_{j=1}^{r-1} a_j(r-1) B(\gamma_j+m-t, t+1)$$

$$= \gamma_1 B\left(m-l+\sum_{i=1}^n \gamma_i, l+1\right) + \sum_{t=0}^{m-l} \binom{m-l}{t} B\left(m-l-t+\sum_{j=1}^n \gamma_j, l+1\right)$$

$$\times \sum_{r=2}^n \sum_{j=1}^{r-1} c_{r-1} a_j(r-1) B(\gamma_j+m-t, t+1)$$

The lemma thus proved. ■

Remark For $m = l$, it follows from the Lemma 2

$$\begin{aligned}
&P\{X_{n+1}, X_{n+2}, \ldots, X_{n+l} \in (X(v-1, n, \tilde{m}, k), X(v, n, \tilde{m}, k)), X_{n+l+1}, X_{n+l+2}, \ldots, X_{n+m} \\
&\quad \notin (X(v-1, n, \tilde{m}, k), X(v, n, \tilde{m}, k))\} \\
&\quad = P\{X_{n+1}, X_{n+2}, \ldots, X_{n+l} \in (X(v-1, n, \tilde{m}, k), X(v, n, \tilde{m}, k))\} \\
&\quad = \gamma_1 B\left(\sum_{i=1}^{n} \gamma_i, l+1\right) + B\left(\sum_{j=1}^{n} \gamma_j, l+1\right) \sum_{r=2}^{n} \sum_{j=1}^{r-1} c_{r-1} a_j(r-1) B(\gamma_j + l, 1) \\
&\quad = B\left(\sum_{i=1}^{n} \gamma_i, l+1\right) \left(\gamma_1 + \sum_{r=2}^{n} \sum_{j=1}^{r-1} c_{r-1} a_j(r-1) B(\gamma_j + l, 1)\right) \\
&\quad = B\left(\sum_{i=1}^{n} \gamma_i, l+1\right) \sum_{r=1}^{n} \sum_{j=1}^{r-1} c_{r-1} a_j(r-1) B(\gamma_j + l, 1) \\
&\quad = B\left(\sum_{i=1}^{n} \gamma_i, l+1\right) \left(\gamma_1 + \sum_{r=2}^{n} (\gamma_r + l) \prod_{j=1}^{r} \frac{\gamma_j}{\gamma_j + l}\right) \\
&\quad = B\left(\sum_{i=1}^{n} \gamma_i, l+1\right) \left(\sum_{r=1}^{n} (\gamma_r + l) \prod_{j=1}^{r} \frac{\gamma_j}{\gamma_j + l}\right).
\end{aligned}$$

Remark For $m = l = 1$, from the Lemma 2 one obtains

$$P\{X_{n+1} \in (X(v-1, n, \tilde{m}, k), X(v, n, \tilde{m}, k))\} = \frac{\sum_{r=1}^{n} (\gamma_r + 1) \prod_{j=1}^{r} \gamma_j / (\gamma_j + 1)}{\left(\sum_{j=1}^{n} \gamma_j\right)\left(1 + \sum_{j=1}^{n} \gamma_j\right)}$$

Remark For $m = l = 1$, $k = 1$, and $\tilde{m} = (0, 0, \ldots, 0)$, for ordinary order statistics we have

$$\begin{aligned}
&P\{X_{n+1} \in (X(v-1, n, 0, 1), X(v, n, 0, 1))\} \\
&\quad = \frac{1}{\left(\sum_{j=1}^{n} (n-j+1)\right)\left(1 + \sum_{j=1}^{n} (n-j+1)\right)} \sum_{r=1}^{n} (n-r+2) \prod_{j=1}^{r} \frac{n-j+1}{n-j+2} \\
&\quad = \frac{4}{3} \frac{n+2}{(n+1)(n^2+n+2)}.
\end{aligned}$$

THEOREM 1 *Let $i = 1, 2, \ldots, m$, and $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$, $\lambda > 0$ and v be the index of the minimal spacing. Denote*

$$\zeta_i = \begin{cases} 1, & X_{n+i} \in (X(v-1, n, \tilde{m}, k), X(v, n, \tilde{m}, k)) \\ 0, & \textit{otherwise} \end{cases}$$

and $T_m = \sum_{i=1}^{m} \zeta_i$ *be the number of those observations that fall into interval* $(X(v-1, n, \tilde{m}, k), X(v, n, \tilde{m}, k))$. *Then the probability function of* T_m *is*

$$P\{T_m = l\} = \binom{m}{l}\gamma_1 B\left(m-l+\sum_{i=1}^{n}\gamma_i, l+1\right) + \binom{m}{l}\sum_{t=0}^{m-l}\binom{m-l}{t}$$
$$\times B\left(m-l-t+\sum_{j=1}^{n}\gamma_j, l+1\right)\sum_{r=2}^{n}\sum_{i=1}^{r-1} c_{r-1}a_i(r-1)B(\gamma_i+m-t, t+1)\cdot$$

Proof By definition of T_m and from the Lemma 2 one can write

$$\begin{aligned}
P\{T_m = l\} &= P\{\text{Exactly } l \text{ of } \zeta_i'\text{s are equal to 1 and } (m-l) \text{ of } \zeta_i'\text{s are equal to 0}\}\\
&= \binom{m}{l} P\{X_{n+1,} X_{n+2,} \ldots, X_{n+l,} \in (X(v-1, n, \tilde{m}, k), X(v, n, \tilde{m}, k)),\\
&\quad \times X_{n+l+1}, X_{n+l+2}, \ldots, X_{n+m} \notin (X(v-1, n, \tilde{m}, k), X(v, n, \tilde{m}, k))\}\\
&= \binom{m}{l}\gamma_1 B\left(m-l+\sum_{i=1}^{n}\gamma_i, l+1\right)\\
&\quad + \binom{m}{l}\sum_{t=0}^{m-l}\binom{m-l}{t} B\left(m-l-t+\sum_{j=1}^{n}\gamma_j, l+1\right)\\
&\quad \times \sum_{r=2}^{n}\sum_{i=1}^{r-1} c_{r-1}a_i(r-1)B(\gamma_i+m-t, t+1)\cdot
\end{aligned}$$

The theorem thus proved. ■

Consider now some special cases.

3. Special case: Progressively type II censored order statistics

For $k = R_n + 1$ and $\tilde{m} = (R_1, R_2, \ldots, R_{n-1})$, $c_{s-1} = \prod_{i=1}^{s}\left(n-i+1+\sum_{u=i}^{n} R_u\right)$, We have $B\left(m-l+\sum_{i=1}^{n}\gamma_i, l+1\right) = B\left(m-l+\sum_{i=1}^{n}\left(n-i+1+\sum_{u=i}^{n}R_u\right), l+1\right)$, $B\left(m-l-t+\sum_{j=1}^{n}\gamma_j, l+1\right) = B\left(m-l-t+\sum_{j=1}^{n}\left(n-j+1+\sum_{u=j}^{n}R_u\right), l+1\right)$, $B(\gamma_i+m-t, t+1) = B\left(m-t+n-i+1+\sum_{u=i}^{n}R_u, t+1\right)$ and we have

$$a_i(r-1) = \frac{(-1)^{r-1-i}}{\left(\prod_{j=1}^{i-1}\sum_{u=1}^{j}(R_{i-u}+1)\right)\left(\prod_{j=1}^{r-i-1}\sum_{u=1}^{j}(R_{i+u-1}+1)\right)},$$

where $\Pi_{\emptyset} = 1$ and $1 \le i \le r-1$.

Let

$$\Psi_a^{(1)} = \prod_{j=1}^{a}\sum_{u=1}^{j}(R_{a+1-u}+1)$$

and

$$\Psi_a^{(2)} = \prod_{j=1}^{a}\sum_{u=1}^{j}(R_{r-a-2+u}+1)$$

where $a \in \{0, 1, 2, \ldots\}$ and $\Pi_{\emptyset} = 1$.

Finally we obtain

$$a_i(r-1) = \frac{(-1)^{r-1-j}}{\Psi_{i-1}^{(1)}\Psi_{r-1-i}^{(2)}}$$

and for Progressively Type II censored order statistics Theorem 2 can be expressed as follows

$$\begin{aligned}
P\{T_m = l\} = &\binom{m}{l}\left(n+\sum_{u=1}^{n}R_u\right)B\left(m-l+\sum_{i=1}^{n}\left(n-i+1+\sum_{u=i}^{n}R_u\right), l+1\right)\\
&+\binom{m}{l}\sum_{t=0}^{m-l}\left[\binom{m-l}{t}B\left(m-l-t+\sum_{j=1}^{n}\left(n-j+1+\sum_{u=j}^{n}R_u\right), l+1\right)\right.\\
&\times\sum_{r=2}^{n}\sum_{i=1}^{r-1}\prod_{j=1}^{r}\left(\left(n-j+1+\sum_{u=j}^{n}R_u\right)\frac{(-1)^{r-1-i}}{\Psi_{i-1}^{(1)}\Psi_{r-1-i}^{(2)}}\right.\\
&\left.\left.\times\ B\left(m-t+n-i+1+\sum_{u=i}^{n}R_u, t+1\right)\right)\right].
\end{aligned}$$

where $\Psi_a^{(1)} = \prod_{j=1}^{a}\sum_{u=1}^{j}(R_{a+1-u}+1)$ and $\Psi_a^{(2)} = \prod_{j=1}^{a}\sum_{u=1}^{j}(R_{r-a-2+u}+1)$.

4. Special case: Ordinary order statistics

If $m = l = 1$ and $\widetilde{\mathbf{R}} = (0, 0, \ldots, 0)$, then we have ordinary order statistics. For this case $\Psi_a^{(1)} = a!$, $\Psi_a^{(2)} = a!$ and

$$\begin{aligned}
P\{T_1 = 1\} &= P\{X_{n+1} \in (X_{v-1:n}, X_{v:n})\}\\
&= nB\left(\sum_{i=1}^{n}(n-i+1), 2\right) + B\left(\sum_{i=1}^{n}(n-i+1), 2\right)\\
&\quad\times\sum_{r=2}^{n}\sum_{i=1}^{r-1}\left(\left(\prod_{j=1}^{r}(n-j+1)\right)\frac{(-1)^{r-1-i}B(n-i+2,1)}{(i-1)!(r-1-i)!}\right).
\end{aligned}$$

Then we have

$$P\{X_{n+1} \in (X_{v-1:n}, X_{v:n})\} = \frac{4n(n+2)}{3(n^2+n+2)n(n+1)}$$

which agrees with the Corollary 1 of Bairamov and Eryılmaz [15].

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