$\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. Denote $c_{r-1} = \prod_{i=1}^{r} \gamma_i$, $r = 1, 2, \ldots, n-1$ and $\gamma_n = k$. The distribution theory of generalized order statistics are given in Kamps and Cramer (2001). Let $a_j(r) = \prod_{i=1, i\neq j}^{r} (\frac{1}{\gamma_i - \gamma_j})$ for $1 \le j \le r \le n$, $a_i^{(r)}(s) = \prod_{j=r+1, i\neq j}^{s} \frac{1}{\gamma_j - \gamma_i}$ for $r+1 \le i \le s \le n$, and $\prod_{\varnothing} = 1$. The probability density function (pdf) of the rth generalized order statistic $X(r, n, \tilde{m}, k)$ is

$$f^{X(r,n,\tilde{m},k)}(x) = c_{r-1} f(x) \sum_{j=1}^{r} a_j(r)(1 - F(x))^{\gamma_j - 1} \quad \text{for } 1 \le r \le n.$$

The cumulative distribution function (cdf) of $X(r, n, \tilde{m}, k)$ is

$$F^{X(r,n,\tilde{m},k)}(x) = 1 - c_{r-1} \sum_{j=1}^{r} \frac{a_j(r)}{\gamma_j} (1 - F(x))^{\gamma_j}.$$

The joint pdf of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$ is

$$f^{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x_r, x_s) = c_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^{(r)}(s) (1 - F(x_r))^{\gamma_j} \left(\frac{1 - F(x_s)}{1 - F(x_r)} \right)^{\gamma_i} \times \frac{f(x_r)}{1 - F(x_r)} \frac{f(x_s)}{1 - F(x_s)}$$

for $1 \le r < s \le n$ and $x_r \le x_s$.

There not exists yet a natural interpretation of generalized order statistics in terms of observed random samples, but an interesting special case is the progressive Type-II censored order statistics. This model is one of the most applicable general models of ordered random variables and is useful in reliability and life time studies. Let $X_1, X_2, \ldots, X_N$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s) representing failure times of n identical units placed on a life test. Under the progressive Type-II right censoring scheme, at the time of ith failure R_i ($i = 1, 2, \ldots, n$ and $n \le N$) surviving items are removed at random from the experiment, where $n + \sum_{i=1}^{n} R_i = N$. Let $\mathbf{R} = (R_1, R_2, \ldots, R_n)$. Denote the n ordered observed failure times by $X_{1:n:N}^{(\mathbf{R})}$, $X_{2:n:N}^{(\mathbf{R})}, \ldots,$ $X_{n:n:N}^{(\mathbf{R})}$. These random variables are called progressive Type-II right censored order statistics from a sample $X_1, X_2, \ldots, X_N$ with progressive censoring scheme $\mathbf{R} = (R_1, R_2, \ldots, R_n)$. A nice description of details of the theory, methods and applications of progressive censoring can be found in Balakrishnan and Aggarwala (2000). If the failure times of the N items originally on the test are from a continuous population with distribution function (df) F and probability density function f, the joint pdf of all n progressively Type-II censored order statistics is

$$f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = c \sum_{i=1}^{n} f(x_i)\{1 - F(x_i)\}^{R_i}, \quad x_1 < x_2 < \cdots < x_n,$$

where $c = N(N - R_1 - 1) \cdots (N - R_1 - R_2 - \cdots - R_{n-1} - n + 1)$. It is not difficult to observe that this is a special case of (1) for $k = R_n + 1$, $m_i = R_i$, $i = 1, 2, \ldots, m-1$ and $\gamma_r = N - \sum_{i=1}^{r-1} R_i - r + 1$, $r = 2, 3, \ldots, n-1$ and $\gamma_1 = N$.

Another choice of parameters in (1) gives a well studied in the literature model of record values. For $m_i = -1$, $i = 1, 2, \ldots, n-1$, $k = 1$ we have the joint pdf

$$f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) = r(x_1)r(x_2)\cdots r(x_{n-1})f(x_n)$$
$$\text{for} -\infty < x_1 < x_2 < \cdots < x_{n-1} < x_n < \infty$$

where $r(x) = \frac{f(x)}{1-F(x)}$, for $0 < F(x) < 1$. The random variables having this joint pdf are record values that can be described as follows. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed (i.i.d) r.v.'s with continuous distribution function F. Define a record times of this sequence as follows: $u(1) = 1, u(n) = \min\{j : j > u(n-1), X_j > X_{u(n-1)}\}, n > 1.$, Let $X_{u(1)}, X_{u(2)}, \ldots$ be corresponding record values.

The details of the theory of records can be found in Galambos (1978), Nagaraja (1988), Nevzorov (1988), Ahsanullah (1995), and Arnold et al. (1998), among others.

There are also other special cases of generalized order statistics such as sequential order statistics, *k*th records, Pfiefer's record model, etc.

1.2. *Exceedances*

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random sample of size n from an absolutely continuous distribution with a common cumulative distribution function (cdf) $F_X(\cdot)$. Denote the ordinary order statistics of $X_1, X_2, \ldots, X_n$ by $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$. Let X_{n+1}, $X_{n+2}, \ldots, X_{n+m}$ be another sample of size m from the same distribution independent from X. It is well known that for $1 \leq i < j \leq n$

$$P\{X_{n+1} \in (X_{i:n}, X_{j:n})\} = \frac{j-i}{n+1}, \tag{2}$$

i.e., $(X_{i:n}, X_{j:n})$ is a distribution free confidence interval containing the future observation in the class of all absolutely continuous distribution functions $\Im_c$. Let $X_1, X_2, \ldots, X_n, X_{n+1}, \ldots, X_{n+m}$ be the sample from the distribution with continuous df F. Denote

$$\xi_k = \begin{cases} 1, & \text{if } X_{i:n} \leq X_{n+k} \leq X_{j:n} \\ 0, & \text{otherwise} \end{cases} \quad k = 1, 2, \ldots, m$$

and

$$S_m = \sum_{i=0}^{m} \xi_i.$$

It is clear that the exceedance statistic S_m is the number of observations X_{n+1}, $X_{n+2}, \ldots, X_{n+m}$ falling into random interval $(X_{i:n}, X_{j:n})$. The distribution of S_m is

$$P\{S_m = k\} = \frac{j-i}{j-i+k}\frac{\binom{m}{k}\binom{n}{j-i}}{\binom{n+m}{k+j-i}}, \quad k = 0, 1, 2, \ldots, m \tag{3}$$

and

$$P\{X_{n+1}, X_{n+2}, \ldots, X_{n+m} \in (X_{i:n}, X_{j:n})\} = \frac{n!(m+j-i-1)!}{(j-i-1)!(m+n)!}$$

(see e.g., Bairamov and Petunin, 1990; Johnson and Kotz, 1991; Matveychuk and Petunin, 1990/1991). The asymptotic distribution of S_m is

$$\lim_{m\to\infty} \sup_{0\leq x\leq 1} \left| P\left\{\frac{S_m}{m} \leq x\right\} - I_x(j-i, n-j+i+1) \right| = 0,$$

where

$$I_x(j-i, n-j+i+1) = \frac{1}{B(j-i, n-j+i+1)} \int_0^x t^{j-i-1}(1-t)^{n-j+i} dt,$$

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt.$$

Bairamov (1997) considered the exceedance statistics in record model. Let $X_{U(n)}$ be nth record and $X_{U(n)+1}$ is the next observation coming after $X_{U(n)}$. For any $k = 1, 2, \ldots, m$ and $r = 1, 2, \ldots$ it is true that

$$P\{X_{U(n)+1} < X_{U(r)}\} = 1 - \frac{1}{2^r}.$$

and

$$P\{X_{U(r)} < X_{U(n)+1} < X_{U(s)}\} = \frac{1}{2^r} - \frac{1}{2^s}.$$

It is clear that $X_{U(n)}$ and the next m observations $X_{U(n)+1}, X_{U(n)+2}, \ldots, X_{U(n)+m}$ coming after $X_{U(n)}$ are independent.

Let us define the following r.v. for given r:

$$\xi_i(r) = 1 \quad \text{if } X_{U(r)+1} < X_{U(r)} \quad \text{and} \quad \xi_i(r) = 0 \quad \text{if } X_{U(r)+1} \geq X_{U(r)}, \quad i = 1, 2, \ldots, m$$

and denote $S_m(r) = \sum_{i=1}^m \xi_i(r)$. It is clear that $S_m(r)$ is the number of observations $X_{U(n)+1}, X_{U(n)+2}, \ldots, X_{U(n)+m}$ which are less than $X_{U(r)}$. Note that the random variables $\xi_1(r), \xi_2(r), \ldots, \xi_m(r)$ are dependent as well as the random variables $\xi_1, \xi_2, \ldots, \xi_m$, above. Bairamov (1997) derived the finite and asymptotic distributions of S_m. In particular, for any $m, r = 1, 2, \ldots$

$$\begin{aligned} P\{S_m(r) = k\} &= \frac{\binom{m}{k}}{(r-1)!} \int_0^\infty e^{-z(m-k+1)}(1-e^{-z})^k z^{r-1} dz \\ &= \binom{m}{k} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^{k-l}}{(m-l+1)^r}. \quad k = 0, 1, 2, \ldots, m. \end{aligned}$$

Wesolowski and Ahsanullah (1998) considered more general models of exceedances based on records of two independent sequences $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ with distribution functions F_X and F_Y, respectively. They apply their results for characterizing equidistributions. Bairamov and Eryilmaz (2000) consider the exceedance model based on spacing having minimal length. Stepanov (2004) considered the multiple exceedance statistics in a record model and derived the joint distributions.

In this article we consider the generalized order statistics $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ based on the continuous distribution function F and observations $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ from the population having distribution function F. For $m_i = 0, i = 1, 2, \ldots, n-1$ the random variables $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are ordinary order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ of the sample $X_1, X_2, \ldots, X_n$ and we assume that $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ are new observations independent from $X_1, X_2, \ldots, X_n$; for $k = R_n + 1$, $m_i = R_i$, $i = 1, 2, \ldots, m-1$ and $\gamma_r = N - \sum_{i=1}^{r-1} R_i - r$, $r = 2, 3, \ldots, n-1$ and $\gamma_1 = N$ the $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are progressive Type-II censored order statistics based on the sample $X_1, X_2, \ldots, X_n$ with distribution function F and we assume that $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ are new observations from F independent from $X_1, X_2, \ldots, X_n$; for $m_i = -1$, $i = 1, 2, \ldots, n-1$, $k = 1$ the random variables $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are records $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$ of the sequence $X_1, X_2, \ldots, X_n, \ldots$ and we assume that $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ are new observations $X_{U(n)+1}, X_{U(n)+2}, \ldots, X_{U(n)+m}$ coming after the rth record, etc.

2. Exceedances in Generalized Order Statistics Models

Let $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ are generalized order statistics based on continuous df F. with pdf f. Denote

$$a_j(r) = \prod_{i=1, i \neq j}^{r} \left(\frac{1}{\gamma_i - \gamma_j} \right)$$

for $1 \leq j \leq r \leq n$,

$$a_i^{(r)}(s) = \prod_{i=r+1, i \neq j}^{s} \frac{1}{\gamma_j - \gamma_i}$$

for $r+1 \leq i \leq s \leq n$, $\prod_{\emptyset} = 1$, $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in R^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, $\gamma_r = k + n - r + M_r > 0$, $c_{r-1} = \prod_{j=1}^{r} \gamma_j$, $r = 1, 2, \ldots, n-1$, and $\gamma_n = k$.

Lemma 2.1. *Let $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ are m i.i.d. random variables with distribution function F. We assume that generalized order statistics $X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)$ based on F and $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ are independent. Then, for $1 \leq r < s \leq n$,*

$$\begin{aligned} P_{r,s}(l, m) :&= P\{X_{n+1}, X_{n+2}, \ldots, X_{n+l} \in (X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k)), \\ &\qquad X_{n+l+1}, X_{n+l+2}, \ldots, X_{n+m} \notin (X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k)t)\} \\ &= c_{s-1} \sum_{t=0}^{m-l} \binom{m-l}{t} \left(\sum_{j=1}^{r} a_j(r) B(t+1, \gamma_j + m - t) \right) \\ &\qquad \times \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) B(\gamma_i + m - l - t, l + 1) \right). \end{aligned} \tag{4}$$

Proof. To avoid cumbersome expressions, denote $X_{n+j} = X_j$, $X(r, n, \tilde{m}, k) = Y_r$, $X(s, n, \tilde{m}, k) = Y_s$, $(-\infty, Y_r] = I_r$, $[Y_s, \infty) = I_s$, and $(Y_r, Y_s) = I_{r,s}$. It is clear that,

$$\begin{aligned}
&P\{X_1, X_2, \dots, X_l \in I_{r,s}, X_{l+1}, X_{l+2}, \dots, X_m \notin I_{r,s}\} \\
&= \sum_{t=0}^{m-l} \binom{m-l}{t} P\{X_1, X_2, \dots, X_l \in I_{r,s} \text{ and exactly } t \text{ from } X_{l+1}, \dots, X_m \\
&\qquad \text{belong to } I_r \text{ and } (m-l-t) \text{ of } X_{l+1}, \dots, X_m \text{ belong to } I_s\} \\
&= \sum_{t=0}^{m-l} \binom{m-l}{t} \int_{-\infty}^{\infty} \int_{y_r}^{\infty} (F(y_s) - F(y_r))^l (F(y_r))^t \\
&\qquad \times (1 - F(y_s))^{m-l-t} f^{Y_r, Y_s}(y_r, y_s) dy_s\, dy_r \\
&= c_{s-1} \sum_{t=0}^{m-l} \binom{m-l}{t} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^{(r)}(s) \int_{-\infty}^{\infty} \int_{y_r}^{\infty} (F(y_s) - F(y_r))^l (F(y_r))^t \\
&\qquad \times (1 - F(y_r))^{\gamma_j - \gamma_i - 1} (1 - F(y_s))^{\gamma_i + m - l - t - 1} dF(y_s)\, dF(y_r) \\
&= c_{s-1} \sum_{t=0}^{m-l} \binom{m-l}{t} \left(\sum_{j=1}^{r} a_j(r) B(t+1, \gamma_j + m - t) \right) \\
&\qquad \times \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) B(\gamma_i + m - l - t, l+1) \right).
\end{aligned}$$

Thus the lemma proved.

Consider now special cases for which the formula (4) has simple form.

1. For $m = l$, we have

$$\begin{aligned}
P_{r,s}(m, m) &= P\{X_{n+1}, X_{n+2}, \dots, X_{n+m} \in (X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k))\} \\
&= c_{s-1} \left(\sum_{j=1}^{r} a_j(r) B(1, \gamma_j + l) \right) \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) B(\gamma_i, l+1) \right) \\
&= c_{s-1} \left(\sum_{j=1}^{r} \frac{a_j(r)}{\gamma_j + l} \right) \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) B(\gamma_i, l+1) \right).
\end{aligned}$$

2. For $m = l = 1$, from Lemma 2.1 we have

$$\begin{aligned}
P_{r,s}(1, 1) &= P\{X_{n+1} \in (X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k))\} \\
&= c_{s-1} \left(\sum_{j=1}^{r} a_j(r) B(1, \gamma_j + 1) \right) \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) B(\gamma_i, 2) \right) \\
&= c_{s-1} \left(\sum_{j=1}^{r} \frac{a_j(r)}{\gamma_j + 1} \right) \left(\sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_i(\gamma_i + 1)} \right) \\
&= c_{s-1} \frac{1}{\prod_{j=1}^{r} (\gamma_j + 1)} \left(\sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_i(\gamma_i + 1)} \right) \qquad (5)
\end{aligned}$$

3. For $m = l = 1$, $k = 1$, and $\tilde{m} = (0, 0, \dots, 0)$, from (4) we have

$$\begin{aligned} c_{s-1} &= \frac{n!}{(n-s)!}, \sum_{j=1}^{r} \frac{a_j(r)}{\gamma_j + 1} = \sum_{j=1}^{r} \frac{(-1)^{r-j}}{(r-j)!(j-1)!(n-j+2)} \\ &= \frac{(-1)^{r-1}}{(r-1)!0!(n+1)} + \frac{(-1)^{r-2}}{(r-2)!1!n} + \cdots + \frac{(-1)^0}{0!(r-1)!(n-r+2)} \\ &= -\frac{(-1)^{2r-1}(r-1)!(n-r+1)!}{(r-1)!(n+1)!} = \frac{(n-r+1)!}{(n+1)!} \end{aligned} \tag{6}$$

and

$$\begin{aligned} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_i(\gamma_i + 1)} &= \sum_{i=r+1}^{s} \frac{(-1)^{s-i}}{(s-i)!(i-r-1)!(n-i+2)(n-i+1)} \\ &= \frac{(-1)^{s-r-1}}{(s-r-1)!0!(n-r+1)(n-r)} + \frac{(-1)^{s-r-2}}{(s-r-2)!1!(n-r)(n-r-1)} \\ &\quad + \cdots + \frac{(-1)^0}{0!(s-r-1)!(n-s+2)(n-s+1)} = \frac{(n-s)!}{(n-r+1)!}(s-r). \end{aligned} \tag{7}$$

Thus, taking into account (6) and (7) in (5) we obtain

$$\begin{aligned} P\{X_{n+1} \in (X(r,n,0,1), X(s,n,0,1))\} &= \frac{n!}{(n-s)!} \frac{(n-r+1)!}{(n+1)!} \frac{(n-s)!}{(n-r+1)!}(s-r) \\ &= \frac{s-r}{n+1}, \end{aligned}$$

which agrees with the formula (2), when the generalized order statistics $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$ are ordinary order statistics $X_{r:n}$ and $X_{s:n}$, respectively.

Now define the following random variables

$$\xi_i = \begin{cases} 1, & X_{n+i} \in (X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k)), \\ 0, & \text{otherwise} \end{cases}$$

$i = 1, 2 \dots, m$ and let $S_m = \sum_{i=1}^{m} \xi_i$. It is clear that S_m represents the number of those $X_{n+1}, X_{n+2}, \dots, X_{n+m}$ random variables that fall into interval $(X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k))$.

Theorem 2.1. *The probability function of exceedance statistic S_m is*

$$\begin{aligned} P\{S_m = l\} &= \binom{m}{l} c_{s-1} \sum_{t=0}^{m-l} \binom{m-l}{t} \left(\sum_{j=1}^{r} a_j(r) B(t+1, \gamma_j + m - t) \right) \\ &\quad \times \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) B(\gamma_i + m - l - t, l + 1) \right), \quad l = 0, 1, \dots, m. \end{aligned} \tag{8}$$

Proof. From the definition of S_m and from the Lemma 2.1 one can write

$$\begin{aligned}
P\{S_m = l\} &= \sum_{(i_1,\ldots,i_m)} P\{\xi_{i_1} = 1, \ldots, \xi_{i_l} = 1, \xi_{i_{l+1}} = 0, \ldots, \xi_{i_m} = 0\} \\
&= \binom{m}{l} P\{X_{n+1,} X_{n+2,} \ldots, X_{n+l,} \in (X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k)), \\
&\qquad X_{n+l+1}, X_{n+l+2}, \ldots, X_{n+m} \notin (X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k))\} \\
&= \binom{m}{l} c_{s-1} \sum_{t=0}^{m-l} \binom{m-l}{t} \left(\sum_{j=1}^{r} a_j(r) B(t+1, \gamma_j + m - t) \right) \\
&\quad \times \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) B(\gamma_i + m - l - t, l + 1) \right)
\end{aligned}$$

□

Thus the theorem proved.

Corollary 2.1. *For $k = 1$, $m_i = 0, i = 1, 2, \ldots, n-1$, and $\gamma_i = n + 1 - i$ the generalized order statistics $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$ are ordinary rth and sth order statistics of the sample $X_1, X_2, \ldots, X_n$ from population with df F and $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ are the new observations from F coming after $X_1, X_2, \ldots, X_n$. For this case* (8) *coincides with the formula* (3).

Corollary 2.2. *For $k = R_n + 1, m_i = R_i,\ i = 1, 2, \ldots, m-1$, and $\gamma_r = N - \sum_{i=1}^{r-1} R_i - r + 1,\ r = 2, 3, \ldots, n-1$ and $\gamma_1 = N$ the generalized order statistics $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$ are progressive Type-II censored rth and sth progressive Type-II censored order statistics $X_{r:n:N}^{(\mathbf{R})}$ and $X_{s:n:N}^{(\mathbf{R})}$ with censoring scheme $(R_1, R_2, \ldots, R_n)$ and in this case*

$$\begin{aligned}
c_{s-1} &= \prod_{i=1}^{s} \left(n - i + 1 + \sum_{u=i}^{n} R_u \right), \\
B(t+1, \gamma_j + m - t) &= B\left(t + 1, n + m - j - t + 1 + \sum_{u=j}^{n} R_u \right), \\
B(\gamma_i + m - l - t, l + 1) &= B\left(n + m - i - t + 1 - l + \sum_{u=i}^{n} R_u, l + 1 \right).
\end{aligned}$$

Then,

$$\begin{aligned}
P\{S_m = l\} &= \binom{m}{l} \prod_{i=1}^{s} \left(n - i + 1 + \sum_{u=i}^{n} R_u \right) \\
&\quad \times \sum_{t=0}^{m-l} \left[\binom{m-l}{t} \left(\sum_{j=1}^{r} \frac{(-1)^{r-j} B(t+1, n+m-j-t+1+\sum_{u=j}^{n} R_u)}{\Psi_{j-1}^{(1)} \Psi_{r-j}^{(2)}} \right) \right. \\
&\quad \left. \times \left(\sum_{i=r+1}^{s} \frac{(-1)^{s-i} B(n+m-i-t+1-l+\sum_{u=i}^{n} R_u, l+1)}{\Psi_{i-r-1}^{(1)} \Psi_{s-i}^{(2)}} \right) \right]. \qquad (9)
\end{aligned}$$

where $\Psi_a^{(1)} = \prod_{j=1}^{a} \sum_{u=1}^{j} (R_{a+1-u} + 1)$ and $\Psi_a^{(2)} = \prod_{j=1}^{a} \sum_{u=1}^{j} (R_{r-a-1+u} + 1)$.

Proof. By definition we have

$$a_j(r) = \prod_{i=1, i\neq j}^{r} \left(j - i + \sum_{u=i}^{n} R_u - \sum_{u=j}^{n} R_u \right)^{-1}$$
$$= \frac{(-1)^{r-j}}{\left(\prod_{i=1}^{j-1} \sum_{u=1}^{i} (R_{j-u} + 1)\right)\left(\prod_{i=1}^{r-j} \sum_{u=1}^{i} (R_{j+u-1} + 1)\right)}$$

where $\Pi_{\emptyset} = 1$ and $1 \leq j \leq r$.

For the constant $a_j(s)$ we have

$$a_i^{(r)}(s) = \prod_{j=r+1, j\neq i}^{a} \left(i - j + \sum_{u=j}^{n} R_u - \sum_{u=i}^{n} R_u \right)^{-1}$$
$$= \frac{(-1)^{s-i}}{\left(\prod_{j=1}^{i-r-1} \sum_{u=1}^{j} (R_{i-u} + 1)\right)\left(\prod_{j=1}^{s-i} \sum_{u=1}^{j} (R_{i+u-1} + 1)\right)},$$

where $\Pi_{\emptyset} = 1$ and $r + 1 \leq i \leq s$.

Denote

$$\Psi_a^{(1)} = \prod_{j=1}^{a} \sum_{u=1}^{j} (R_{a+1-u} + 1), \quad \Psi_a^{(2)} = \prod_{j=1}^{a} \sum_{u=1}^{j} (R_{r-a-1+u} + 1)$$

where $a \in \{0, 1, 2, \ldots\}$ and $\Pi_{\emptyset} = 1$.

Then one can write

$$a_j(r) = \frac{(-1)^{r-j}}{\Psi_{j-1}^{(1)} \Psi_{r-j}^{(2)}}, \quad a_i^{(r)}(s) = \frac{(-1)^{s-i}}{\Psi_{i-r-1}^{(1)} \Psi_{s-i}^{(2)}}.$$

and from Theorem 2.1 we obtain (9). □

Remark 2.1. If $m = l = 1$ and $\widetilde{\mathbf{R}} = (0, 0, \ldots, 0)$, then, $\Psi_a^{(1)} = a!$ and $\Psi_a^{(2)} = a!$. Then from (9) for ordinary order statistics one has

$$P\{S_1 = 1\} = P\{X_{n+1} \in (X_{r:n}, X_{s:n})\}$$
$$= \left(\prod_{j=1}^{s} (n - i + 1) \right) \left(\sum_{j=1}^{r} \frac{(-1)^{r-j} B(1, n - j + 2)}{(j - 1)!(r - j)!} \right)$$
$$\times \left(\sum_{i=r+1}^{s} \frac{(-1)^{s-i} B(n - i + 1, 2)}{(i - r - 1)!(s - i)!} \right)$$
$$= \frac{n!}{(n - s)!} \left(\sum_{j=1}^{r} \frac{(-1)^{r-j} B(1, n - j + 2)}{(j - 1)!(r - j)!} \right) \left(\sum_{i=r+1}^{s} \frac{(-1)^{s-i} B(n - i + 1, 2)}{(i - r - 1)!(s - i)!} \right)$$
$$= \frac{s - r}{n + 1},$$

which agrees with the well-known formula (2).

Table 1
Progressively censored sample generated from times to breakdown data on insulating fluid tested at 34 kV by Nelson (1982)

i	1	2	3	4	5	6	7	8
$X^{\tilde{R}}_{i:8:19}$	0.19	0.78	0.96	1.31	2.78	4.85	6.5	7.35
R_i	0	0	3	0	3	0	0	5

The following example is useful for numerical illustration of the result of Theorem 2.1 for important special case when generalized order statistics $X(r, n, \tilde{m}, k) = X^{(\mathbf{R})}_{r:n:N}$ and $X(s, n, \tilde{m}, k) = X^{(\mathbf{R})}_{s:n:N}$, $r < s$, and $\mathbf{R} = (R_1, R_2, \ldots, R_n)$.

Example 2.1 (Exceedance Statistics for Progressively Type-II Censored Order Statistics)**.** Nelson (1982, p. 105, Table 1.1) presents the results of a life-test experiment in which specimens of a type of electrical insulating fluid were subject to a constant voltage stress. The length of time until each specimen failed (or breakdown) was observed. (Table 1) (see also Fernandez, 2004).

We provide the probability mass functions of the exceedance statistics S_3 for different censoring schemes (Table 2).

Table 2
The probability mass functions of the exceedance statistics for different censoring schemes

N	n	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	l	$P\{S_3 = l\}$
19	8	0	0	3	0	3	0	0	5	0	0.4434
										1	0.3818
										2	0.1492
										3	0.0256
19	8	3	3	0	0	0	0	0	5	0	0.3769
										1	0.3952
										2	0.1884
										3	0.0395
19	8	3	0	3	0	0	0	0	5	0	0.3925
										1	0.3927
										2	0.1789
										3	0.0359
19	8	2	0	2	1	0	2	0	4	0	0.4293
										1	0.3861
										2	0.1568
										3	0.0278
19	8	2	0	2	1	2	1	1	2	0	0.4040
										1	0.3899
										2	0.1723
										3	0.0338

3. Asymptotic Distributions

In this section we investigate asymptotic behavior of the exceedance statistics S_m and T_m. The following lemma is essential for further results.

Lemma 3.1. *Let $X_1, X_2, \dots, X_m$ be iid random variables with distribution function F. Assume that Z_1 and Z_2 are two continuous random variables with joint distribution function $F_{Z_1,Z_2}(t, s)$ such that $Z_1 \leq Z_2$ a.s. and X, Z_1 and $_2$ have common support R. Denote*

$$v(m) = \#\{X_i : X_i \in (Z_1, Z_2)\}.$$

Then

$$\limsup_{m\to\infty} \left| P\left\{ \frac{v(m)}{m} \leq x \right\} - P\{F(Z_2) - F(Z_1) \leq x\} \right| = 0.$$

The proof is similar to the proof of the Lemma 3.1 of Bairamov and Eryilmaz (2000).

The following theorems are consequences of the Lemma 3.1.

Theorem 3.1. *The asymptotic distribution of S_m/m for large m is*

$$\lim_{n\to\infty} \sup_{0\leq x\leq 1} \left| P\left\{ \frac{S_m}{m} \leq x \right\} - P\{F(X(s, n, \tilde{m}, k) - F(X(r, n, \tilde{m}, k)) \leq x\} \right| = 0.$$

Theorem 3.2. *The asymptotic distribution of T_m/m for large m is*

$$\lim_{n\to\infty} \sup_{0\leq x\leq 1} \left| P\left\{ \frac{T_m}{m} \leq x \right\} - P\{F(X(v, n, \tilde{m}, k) - F(X(v-1, n, \tilde{m}, k)) \leq x)\} \right| = 0,$$

where v is the index of spacing having minimal length and $T_m = \sum_{i=1}^{m} \zeta_i$, and

$$\zeta_i = \begin{cases} 1, & X_{n+i} \in (X(v, n, \tilde{m}, k), X(v-1, n, \tilde{m}, k)) \\ 0, & \textit{otherwise} \end{cases}, \quad i = 1, 2, \dots, m.$$

For ordinary order statistics $X_{r:n}$ and $X_{s:n}$ the limiting distribution function $P\{F(X_{s:n}) - F(X_{r:n}) \leq x\}$ is

$$P\{F(X_{s:n}) - F(X_{r:n}) \leq x\} = \frac{1}{B(s-r, n-s+r-1)} \int_{-\infty}^{x} t^{s-r-1}(1-t)^{n-s+r-1} dt,$$

which is a *Beta*$(s-r, n-s+r-1)$ distribution. For record values $X_{U(r)}$ and $X_{U(s)}$, the limiting df is

$$P\{F(X_{U(s)}) - F(X_{U(r)}) \leq x\} = P\{U_{rs} \leq x\},$$

where $U_{rs} = F(X_{u(s)}) - F(X_{u(r)})$ has df

$$P\{U_{rs} \le x\} = \frac{1}{(r-1)!(s-r-1)!} \int_0^x \int_0^{1-t_1} \left[\ln \frac{1}{1-t_2}\right]^{r-1} \frac{1}{1-t_2} \times \left[\ln \frac{1-t_1}{1-t_1-t_2}\right]^{s-r-1} dt_2\, dt_1,$$

and pdf

$$f_{U_{rs}}(x) = \frac{1}{(r-1)!(s-r-1)!} \int_0^{1-x} \left[\ln \frac{1}{1-t_2}\right]^{r-1} \times \frac{1}{1-t_2}\left[\ln \frac{1-x}{1-x-t_2}\right]^{s-r-1} dt_2, \quad 0 < x < 1.$$

Furthermore, $U_{k-1,k} = F(X_{u(k)}) - F(X_{u(k-1)})$ has df (see Bairamov and Eryilmaz, 2000)

$$D_k(x) \equiv P\{U_{k-1,k} \le x\} = \frac{1}{(k-1)!} \int_0^x \left[\ln \frac{1}{u}\right]^{k-1} du, \quad 0 < x < 1$$

and pdf

$$d_k(x) \equiv f_{U_{k-1,k}}(x) = \frac{1}{(k-1)!}\left[\ln \frac{1}{x}\right]^{k-1}, \quad 0 < x < 1.$$

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