Statistics and Probability Letters 80 (2010) 196–205

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Numbers of near-maxima for the bivariate case

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ARTICLE INFO

Article history:
Received 21 August 2009
Received in revised form 27 September 2009
Accepted 13 October 2009
Available online 21 October 2009

MSC:
60G70
62G30

ABSTRACT

Let $\overline{Z}_1 = (X_1, Y_1) \dots, \overline{Z}_n = (X_n, Y_n)$ be independent and identically distributed random vectors with continuous distribution. Let $K_n(a, b_1, b_2)$ be the number of sample elements that belong to the open rectangle $(X_{\max}^{(n)} - a, X_{\max}^{(n)}) \times (Y_{\max}^{(n)} - b_1, Y_{\max}^{(n)} + b_2)$ — numbers of near-maxima in the bivariate case. In the present paper, we discuss asymptotic properties of $K_n(a, b_1, b_2)$ and $K_n(\infty, 0, \infty)$.

1. Introduction

Assume in the following, $\overline{Z} = (X, Y), \overline{Z}_1 = (X_1, Y_1), \dots, \overline{Z}_n = (X_n, Y_n)$ are independent and identically distributed random vectors with continuous distribution $F(x, y)$ and marginal distributions $H(x) = P\{X \le x\}$ and $G(y) = P\{Y \le y\}$. Let

$$\overline{M}_n = \left\{\overline{Z}_i : X_i = \max\{X_1, \dots, X_n\}\right\} \quad (1 \le i \le n).$$

Denote the coordinates of $\overline{M}_n$ as $(X_{\max}^{(n)}, Y_{\max}^{(n)})$. The first coordinate $X_{\max}^{(n)}$ is the maximum among $X_1, \dots, X_n$, and $Y_{\max}^{(n)}$ is the concomitant of this maximum.

For $a > 0, b_1 > 0, b_2 > 0$, let us define

$$K_n(a, b_1, b_2) = \#\{j = 1, 2, \dots, n : \quad \overline{Z}_j \in \Pi\},$$

where Π is the open rectangle $(X_{\max}^{(n)} - a, X_{\max}^{(n)}) \times (Y_{\max}^{(n)} - b_1, Y_{\max}^{(n)} + b_2)$. That way, $K_n(a, b_1, b_2)$, which can take values $0, 1, \dots, n - 1$, is the number of sample observations registered in Π.

Let us also define K_n as

$$K_n = K(\infty, 0, \infty).$$

The variable K_n is the number of sample observations that are registered in the upper left quarter-plane defined by the lines $x = X_{\max}^{(n)}$ and $y = Y_{\max}^{(n)}$.

In the univariate case the asymptotic theory of the numbers of near-maxima is discussed in Pakes and Steutel (1997), Khmaladze et al. (1997), Pakes and Li (1998), Li and Pakes (1998), Li (1999), Pakes (2000, 2005), Hashorva and Hüsler (2000, 2005, 2008), Hashorva (2003), Hu and Su (2003), Balakrishnan and Stepanov (2004, 2005, 2008), Dembinska et al. (2007), Balakrishnan et al. (2009), and Bairamov and Stepanov (under review); see also the references in these papers.

The number of double maxima is investigated in the papers of Hashorva and Hüsler (2001, 2002), and Hashorva (2004). In our paper, we discuss asymptotic properties of $K_n(a, b_1, b_2)$ and K_n.

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doi:10.1016/j.spl.2009.10.007

The paper is organized as follows. In Section 2, we analyze the position of the random point $\overline{M}_n$ when $n \to \infty$. This is important, because knowing this position we are able to put conditions on the neighborhood of $\overline{M}_n$ and derive limit results for $K_n(a, b_1, b_2)$ and K_n. In Section 3, we study distributional and moment properties of $K_n(a, b_1, b_2)$ and K_n. Limiting properties of these random variables are discussed in Sections 4 and 5. In Section 4 we analyze limiting properties when the support is bounded from the right, whereas in Section 5, these properties are investigated when the support is unbounded. In the ends of Sections 4 and 5 illustrative examples are presented. The proof of auxiliary Proposition 5.1 is postponed till Appendix.

The designations $\xrightarrow{d}$, $\xrightarrow{p}$, $\xrightarrow{a.s.}$ stand, in the following, for convergence in distribution, convergence in probability, and almost sure convergence, respectively.

2. Preliminaries

Discussing asymptotic properties of $K_n(a, b_1, b_2)$ and K_n let us first analyze the position of the point $\overline{M}_n$ when $n \to \infty$. In David and Nagaraja (2003), it is shown that

$$P\{X_{\max}^{(n)} \le x, Y_{\max}^{(n)} \le y\} = n \int_{-\infty}^{x} H^{n-1}(u) F(\mathrm{d}u, y).$$

It follows that $P\{X_{\max}^{(n)} \le x\} = H^n(x)$ and $X_{\max}^{(n)} \xrightarrow{a.s.} r_H$, where $r_H = \sup\{x \in R : H(x) < 1\}$ is the right extremity of H.

The limit behavior of $Y_{\max}^{(n)}$ is ruled by the type of dependence between X and Y.

(i) Suppose that for all $y < r_G$ (r_G is the right extremity of G)

$$\lim_{n\to\infty} P\{Y_{\max}^{(n)} \le y\} = \lim_{n\to\infty} n \int_{\mathbb{R}} H^{n-1}(x) F(\mathrm{d}x, y) = 0. \tag{2.1}$$

This means $Y_{\max}^{(n)} \xrightarrow{p} r_G$. In this case, we address to the distribution F as to a large maxima concomitant distribution; see Example 5.1 in this respect.

(ii) Suppose that for some c and any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\{c - \varepsilon < Y_{\max}^{(n)} \le c + \varepsilon\} = \lim_{n\to\infty} n \int_{\mathbb{R}} H^{n-1}(x) \left[F(\mathrm{d}x, c+\varepsilon) - F(\mathrm{d}x, c-\varepsilon)\right] = 1. \tag{2.2}$$

This means $Y_{\max}^{(n)} \xrightarrow{p} c$ (in particular, if $c = r_G$, we have the previous case). Here, we address to F as to a c-stable maxima concomitant distribution. In Example 4.1, Example 5.2 and Example 5.3 such distributions are presented.

(iii) If such a constant c, for which the limit in (2.2) equals one, does not exist, then $Y_{\max}^{(n)}$ converges in probability nowhere as $n \to \infty$. In this case, we address to F as to an unstable maxima concomitant distribution. For example, if $F(x, y) = H(x)G(y)$, i.e. X and Y are independent, then $P\{Y_{\max}^{(n)} \le y\} = G(y)$ and for all c

$$P\{c - \varepsilon < Y_{\max}^{(n)} \le c + \varepsilon\} = G(c+\varepsilon) - G(c-\varepsilon) < 1.$$

In Example 4.2, we also present an unstable maxima concomitant distribution F for which X and Y are dependent.

That way, if X and Y are independent then F is an unstable maxima concomitant distribution. If they are dependent and, in addition, their joint distribution satisfies (2.2) for some c, then F is a c-stable maxima concomitant distribution.

We have just introduced a new notation — the stable maxima concomitant distribution. In our work, we analyze the limit behavior of $K_n(a, b_1, b_2)$ and K_n in these terms.

3. Distributional and moment results

Lemma 3.1. *The distribution of $K_n(a, b_1, b_2)$ can be given in two forms*

$$P\{K_n(a, b_1, b_2) = k\} = n \binom{n-1}{k} \int_{\mathbb{R}^2} P_1^k (H(x) - P_1)^{n-k-1} F(\mathrm{d}x, \mathrm{d}y) \quad (k = 0, \ldots, n-1), \tag{3.1}$$

$$P\{K_n(a, b_1, b_2) \ge k\} = n(n-1) \binom{n-2}{k-1} \int_{\mathbb{R}^2} \int_x^{x+a} \int_{y-b_2}^{y+b_1} P_2^k (H(u) - P_2)^{n-k-1} F(\mathrm{d}u, \mathrm{d}v) F(\mathrm{d}x, \mathrm{d}y)$$
$$(k = 1, \ldots, n-1), \tag{3.2}$$

where

$$P_1 = P_1(x, a, y, b_1, b_2) = F(x, y + b_2) - F(x, y - b_1) - F(x - a, y + b_2) + F(x - a, y - b_1)$$

and

$$P_2 = F(u, v + b_2) - F(u, v - b_1) - F(x, v + b_2) + F(x, v - b_1)$$

are the probabilities that $\overline{Z}$ belongs to the rectangles $(x - a, x) \times (y - b_1, y + b_2)$ and $(x, u) \times (v - b_1, v + b_2)$, respectively.

Proof. We prove only equality (3.2), which is not so obvious.

We suppose that one variable is observed at the point (x, y), and another one, which is $\overline{M}_n$, at the point (u, v), where $x < u < x + a$ and $y - b_2 < v < y + b_1$. The event $\{K_n(a, b_1, b_2) \geq k\}$ now happens if $k - 1$ among the rest $n - 2$ observations are registered in the rectangle $(x, u) \times (v - b_1, v + b_2)$. $\square$

Corollary 3.1. *The distribution of K_n can be derived from (3.1):*

$$P\{K_n = k\} = n \binom{n-1}{k} \int_{\mathbb{R}^2} F^{n-k-1}(x, y)(H(x) - F(x, y))^k F(\mathrm{d}x, \mathrm{d}y) \quad (k = 0, \dots, n - 1). \tag{3.3}$$

It follows from (3.1)–(3.3) that

$$EK_n(a, b_1, b_2) = n(n-1) \int_{\mathbb{R}^2} P_1(H(x) - P_1)^{n-2} F(\mathrm{d}x, \mathrm{d}y),$$

$$EK_n(a, b_1, b_2) = n(n-1) \int_{\mathbb{R}^2} \int_x^{x+a} H^{n-2}(u) \left[F(\mathrm{d}u, y + b_1) - F(\mathrm{d}u, y - b_2)\right] F(\mathrm{d}x, \mathrm{d}y) \tag{3.4}$$

and

$$EK_n = n(n-1) \int_{\mathbb{R}^2} (H(x) - F(x, y)) H^{n-2}(x) F(\mathrm{d}x, \mathrm{d}y). \tag{3.5}$$

4. Limit results when the support is bounded from the right

In this section, we suppose that $r_H < \infty$. If F is a c-stable maxima concomitant distribution, it is natural to expect that for rather large n the random point $\overline{M}_n$ locates near the fixed point (r_H, c). Since that n, and further as n increases, the number of observations registered in the rectangle near $\overline{M}_n$ can be approximated by the Bernoulli law with probability of success P_1. Particular forms of limit results for $K_n(a, b_1, b_2)$ and K_n are given below.

Theorem 4.1. *Let $F(x, y)$ be a c-stable maxima concomitant distribution. Then*

$$\frac{EK_n(a, b_1, b_2)}{n} \to P_1(r_F, a, c, b_1, b_2) \quad (n \to \infty)$$

and

$$\frac{K_n(a, b_1, b_2)}{n} \to_p P_1(r_F, a, c, b_1, b_2) \quad (n \to \infty).$$

Proof. We prove only the first result of this theorem for the case when $c = r_G$. The second result can be obtained similarly by analyzing the probability generating function of $K_n(a, b_1, b_2)$.

By (3.4),

$$\frac{EK_n(a, b_1, b_2)}{n} = I_1 - I_2,$$

where

$$I_i = (n-1) \int_{\mathbb{R}^2} \int_x^{x+a} H^{n-2}(u) F(\mathrm{d}u, y + (-1)^{i-1} b_i) F(\mathrm{d}x, \mathrm{d}y) \quad (i = 1, 2).$$

We have

$$\begin{aligned}
I_1 &= o(1) + (n-1) \int_{\mathbb{R}} \int_{r_G - b_1}^{r_G} \int_x^{x+a} H^{n-2}(u) \mathrm{d}F(u) F(\mathrm{d}x, \mathrm{d}y) \\
&= o(1) + \int_{\mathbb{R}} \int_{r_G - b_1}^{r_G} \left[H^{n-1}(x + a) - H^{n-1}(x)\right] F(\mathrm{d}x, \mathrm{d}y) \\
&= o(1) + \int_{r_H - a}^{r_H} \left[H^{n-1}(x + a) - H^{n-1}(x)\right] \mathrm{d}H(x) - \int_{r_H - a}^{r_H} \left[H^{n-1}(x + a) - H^{n-1}(x)\right] F(\mathrm{d}x, r_G - b_1).
\end{aligned}$$

Then

$$\int_{r_H - a}^{r_H} \left[H^{n-1}(x + a) - H^{n-1}(x)\right] \mathrm{d}F(x) = 1 - H(r_H - a) + o(1).$$

By (2.2),

$$\int_{r_H-a}^{r_H}\left[H^{n-1}(x+a)-H^{n-1}(x)\right]F(\mathrm{d}x, r_G-b_1)=G(r_G-b_1)-F(r_H-a, r_G-b_1)+o(1).$$

Applying similar argument, it is possible to show that

$$I_2=o(1)\quad(n\to\infty).\quad\square$$

The following corollary can be easily obtained from Theorem 4.1.

Corollary 4.1. *Let F be a c-stable concomitant distribution. Then*

$$\frac{EK_n}{n}\to 1-G(c)\quad\text{and}\quad\frac{K_n}{n}\xrightarrow{p}1-G(c)\quad(n\to\infty).$$

The above limit results are illustrated by examples.

Example 4.1. Suppose F is uniform in the square determined by the lines:

$$\begin{aligned}&y+x=1,\quad y+x=-1,\\&y-x=1,\quad y-x=-1.\end{aligned}$$

For $y\geq 0$ this distribution has the form

$$F(x,y)=\begin{cases}\dfrac{(1+x)^2}{2} & y>x+1,-1<x\leq 0,\\[2pt] \dfrac{1+2xy+(1+x)^2-(1-y)^2}{4} & y<x+1,-1<x\leq 0,\\[2pt] \dfrac{3+2xy-(1-x)^2-(1-y)^2}{4} & y<1-x,0<x\leq 1,\\[2pt] \dfrac{3+2xy-(1-x)^2-(1-y)^2-(x+y-1)^2}{4} & y>1-x,0<x\leq 1,\end{cases}$$

where the marginal distribution is

$$H(x)=\begin{cases}\dfrac{(1+x)^2}{2} & -1<x\leq 0,\\[2pt] 1-\dfrac{(1-x)^2}{2} & 0<x\leq 1\end{cases}$$

and the density distribution is

$$F(\mathrm{d}x,y)=\begin{cases}(1+x)\mathrm{d}x & y>x+1,-1<x\leq 0,\\ \dfrac{y+1+x}{2}\mathrm{d}x & y<x+1,-1<x\leq 0,\\ \dfrac{y+1-x}{2}\mathrm{d}x & y<1-x,0<x\leq 1,\\ (1-x)\mathrm{d}x & y>1-x,0<x\leq 1.\end{cases}$$

Then for any $\varepsilon>0$

$$\begin{aligned}n\int_{-1}^{1}H^{n-1}(x)F(\mathrm{d}x,y)&=n\int_{-1}^{1-\varepsilon}H^{n-1}(x)F(\mathrm{d}x,y)+n\int_{1-\varepsilon}^{1}H^{n-1}(x)F(\mathrm{d}x,y)\\&=o(1)+n\int_{1-\varepsilon}^{1}\left(1-\frac{(1-x)^2}{2}\right)^{n-1}(1-x)\mathrm{d}x\\&=o(1)+1-\left(1-\varepsilon^2/2\right)^n.\end{aligned}$$

It follows that

$$n\int_{-1}^{1}H^{n-1}(x)F(\mathrm{d}x,\varepsilon)\to 1.$$

Analyzing F when $y < 0$, it is possible to show that for any $\varepsilon > 0$

$$n \int_{-1}^{1} H^{n-1}(x) F(\mathrm{d}x, -\varepsilon) \to 0.$$

That way, condition (2.2) holds with $c = 0$ and F is a 0-stable maxima concomitant distribution. It follows from Theorem 4.1 that for $0 < b_1 < b_2 < a < 1$ (we do not consider other possible choices of a, b_1, b_2)

$$\frac{EK_n(a, b_1, b_2)}{n} \to \frac{b_1(2a - b_1) + b_2(2a - b_2)}{4}$$

and

$$\frac{K_n(a, b_1, b_2)}{n} \to_p \frac{b_1(2a - b_1) + b_2(2a - b_2)}{4}.$$

It follows from Corollary 4.1 that

$$\frac{EK_n}{n} \to \frac{1}{2}, \quad \frac{K_n}{n} \xrightarrow{p} \frac{1}{2}.$$

Example 4.2. Let

$$F(x, y) = \begin{cases} x & \text{if } x < y, \\ y + y\log(x/y) & \text{if } x \geq y \end{cases} \quad (0 < x < 1, 0 < y < 1)$$

with marginal distribution $H(x) = x$ $(0 < x < 1)$ and density distribution

$$F(\mathrm{d}x, y) = \begin{cases} \mathrm{d}x & \text{if } x < y, \\ \dfrac{y}{x}\mathrm{d}x & \text{if } x \geq y. \end{cases}$$

Then

$$n \int_{\mathbb{R}} H^{n-1}(x) \left[F(\mathrm{d}x, c + \varepsilon) - F(\mathrm{d}x, c - \varepsilon)\right] \to 2\varepsilon$$

for any $c \in [0, 1]$. That way, F is an unstable maxima concomitant distribution and Theorem 4.1 cannot be applied here.

5. Limit results in the case of unbounded support

In this section, we suppose that $r_H = \infty$. In Section 5.1 we discuss the asymptotic behavior of K_n. Limiting properties of $K_n(a, b_1, b_2)$ are studied in Section 5.2. Some examples are proposed in Section 5.3. The basic results of our paper presented in the end of Section 5.2.

5.1. Limit results for K_n

Let F be a c-stable maxima concomitant distribution. Then, the random point $\overline{M}_n$ moves to (∞, c) as n increases. The number of observations registered in the upper semi-plane defined by the line $y = c$ can be approximated by the Bernoulli law with probability of success $1 - G(c)$. Particular forms of limit laws for K_n are given below.

Theorem 5.1. *Let F be c-stable maxima concomitant distribution, where $c < r_G$. Then*

1.
$$\frac{EK_n}{n} \to 1 - G(c),$$

2.
$$\frac{K_n}{n} \xrightarrow{p} 1 - G(c),$$

3.
$$\frac{K_n - n(1 - G(c))}{\sqrt{nG(c)(1 - G(c))}} \xrightarrow{d} \xi_{N(0,1)},$$

where $\xi_{N(0,1)}$ is a random variable having the standard normal law.

Proof. We start with showing the truth of the second and third results of this theorem. By (2.2), the probability generating function of K_n can be written as

$$Es^{K_n} = n\int_{\mathbb{R}^2} (sH(x) + (1-s)F(x,y))^{n-1}\ F(\mathrm{d}x, \mathrm{d}y)$$
$$= n\int_{\mathbb{R}}\int_{c-\varepsilon}^{c+\varepsilon} (sH(x) + (1-s)F(x,y))^{n-1}\ F(\mathrm{d}x, \mathrm{d}y) + o(1).$$

Choose $z \in (c-\varepsilon, c+\varepsilon)$ such that

$$n\int_{\mathbb{R}}\int_{c-\varepsilon}^{c+\varepsilon} (sH(x) + (1-s)F(x,y))^{n-1}\ F(\mathrm{d}x, \mathrm{d}y) = n\int_{\mathbb{R}}\int_{c-\varepsilon}^{c+\varepsilon} (sH(x) + (1-s)F(x,z))^{n-1}\ F(\mathrm{d}x, \mathrm{d}y)$$

Then, for any fixed x_0,

$$Es^{K_n} = n\int_{x_0}\left(s + (1-s)\frac{F(x,z)}{H(x)}\right)^{n-1} H^{n-1}(x)[F(\mathrm{d}x, c+\varepsilon) - F(\mathrm{d}x, c-\varepsilon)] + o(1).$$

Choose $\delta_1, \delta_2 > 0$ such that

$$n\int_{x_0} H^{n-1}(x)[F(\mathrm{d}x, c+\varepsilon) - F(\mathrm{d}x, c-\varepsilon)] < 1 + \delta_1 \quad (n > N)$$

and

$$F(x,z)/H(x) < G(z) + \delta_2 \quad (x > x_0).$$

Then

$$Es^{K_n} < [s + (1-s)(G(z) + \delta_2)]^{n-1}(1+\delta_1) + o(1) \quad (n > N).$$

Similarly, the probability generating function of K_n can be estimated from the below. By the continuity of G,

$$Es^{K_n} = [s + (1-s)G(c)]^{n-1} + o(1). \tag{5.1}$$

The second result of this theorem is now obtained by choosing $s = s_n = s^{1/n}$.

The form of the central limit theorem for K_n follows from the binomial component in (5.1). See also pages 186, 187 in Pakes and Steutel (1997).

Since the bounded convergence in probability implies the convergence of the moments, the first result of Theorem 5.1 also follows. □

5.2. Limit results for $K_n(a, b_1, b_2)$

The following limit

$$\lim_{x\to\infty}\frac{1 - H(x+a)}{1 - H(x)} = \beta(a) \in [0, 1], \tag{5.2}$$

proposed in Pakes and Steutel (1997), is used for distribution tail classification in the univariate theory of near-maxima. If the limit in (5.2) exists, the distribution tail $1-H(x)$ is classified as "thin" if $\beta(a) = 0$, "medium" if $0 < \beta(a) < 1$ and "thick" if $\beta(a) = 1$. Based on this classifications different limit laws are obtained for the number of univariate near-maxima in the case of unbounded support. In particular, it is shown that if the tail $1 - H(x)$ is "medium", then the limiting distribution for the number of near-maxima is geometric.

In the bivariate case similar results hold true.

Theorem 5.2. *Let F be a c-stable maxima concomitant distribution and $c \le r_G < \infty$. Let the limit in (5.2) exist and $\beta(a) > 0$. Suppose, also, the limit*

$$\lim_{x\to\infty}\frac{G(y) - F(x,y)}{1 - H(x)} \tag{5.3}$$

exists for all y. Then

$$K_n(a, b_1, b_2) \xrightarrow{d} Geo(\beta(a)),$$

where $Geo(p)$ is a geometrically distributed random variable with parameter p.

Proof. Let us consider the probability generating function of $K_n(a, b_1, b_2)$

$$Es^{K_n(a,b_1,b_2)} = n \int_{\mathbb{R}^2} [H(x) - (1-s)P_1(x, a, y, b_1, b_2)]^{n-1} F(\mathrm{d}x, \mathrm{d}y).$$

For further analysis of $Es^{K_n(a,b_1,b_2)}$ we need the following auxiliary proposition, which proof is postponed till Appendix.

Proposition 5.1. *Let F be a c-stable maxima concomitant distribution and $c \le r_G < \infty$. Let the limits in (5.2) and (5.3) exist and $\beta(a) > 0$. Then*

$$\lim_{x\to\infty} \frac{P_1(x, a, y, b_1, b_2)}{1 - H(x)} = 0 \quad (y < c - b_2 \text{ or } y > c + b_1)$$

and

$$\lim_{x\to\infty} \frac{P_1(x, a, y, b_1, b_2)}{1 - H(x)} = \frac{1 - \beta(a)}{\beta(a)} \quad (c - b_2 < y < c + b_1),$$

where $b_1 = 0$ if $c = r_G$.

By Proposition 5.1, we can choose $\varepsilon_1,\ \varepsilon_2 > 0$ such that

$$\frac{P_1(x, a, y, b_1, b_2)}{1 - H(x)} < \varepsilon_1 \quad (y < c - b_2 \text{ or } y > c + b_1,\ x > x_0)$$

and

$$\frac{P_1(x, a, y, b_1, b_2)}{1 - H(x)} < \frac{1 - \beta(a) + \varepsilon_2}{\beta(a) - \varepsilon_2} \quad (c - b_2 < y < c + b_1,\ x > x_0).$$

Then

$$\begin{aligned} Es^{K_n(a,b_1,b_2)} > {} & n \int_{x_0}^{\infty} \int_{-\infty}^{c-b_2} [H(x) - (1-s)\varepsilon_1(1 - H(x))]^{n-1} F(\mathrm{d}x, \mathrm{d}y) \\ & + n \int_{x_0}^{\infty} \int_{c+b_1}^{\infty} [H(x) - (1-s)\varepsilon_1(1 - H(x))]^{n-1} F(\mathrm{d}x, \mathrm{d}y) \\ & + n \int_{x_0}^{\infty} \int_{c-b_2}^{c+b_1} \left[H(x) - (1-s)\frac{1 - \beta(a) - \varepsilon_2}{\beta(a) + \varepsilon - \varepsilon_2}(1 - H(x))\right]^{n-1} F(\mathrm{d}x, \mathrm{d}y) + o(1), \end{aligned}$$

where in the case when $c = r_G$ we have only the first and the third integral. It is possible to show that the first two integrals on the right-hand side of the last inequality tend to zero and the third integral tends to

$$\frac{\beta(a) + \varepsilon}{1 - s(1 - \beta(a) - \varepsilon)}.$$

In the same way $Es^{K_n(a,b_1,b_2)}$ can be estimated from the above. We conclude

$$\lim_{n\to\infty} Es^{K_n(a,b_1,b_2)} = \frac{\beta(a)}{1 - s(1 - \beta(a))}.$$

By the form of the probability generating function of $K_n(a, b_1, b_2)$, the result follows. □

Comment 5.1. *It is strange, at first look, that the limiting distribution of $K_n(a, b_1, b_2)$ is free of b_1 and b_2. We may explain it like this. When F is a c-stable maxima concomitant distribution and x is large, the probability mass is concentrated basically near the line $y = c$ and $Y_{\max}^{(n)}$ is close to c. The height of the rectangle $\prod = (X_{\max}^{(n)} - a, X_{\max}^{(n)}) \times (c - b_1, c + b_2)$ is then unimportant for counting the sample observations registered in $\prod$, because these observations (when x is great) are mainly located near the line $y = c$.*

With little modification in the proof of Theorem 5.2, one can obtain the following theorem.

Theorem 5.3. *Let F be a large maxima concomitant distribution and $r_G = \infty$. Let the limit in (5.2) exist and $\beta(a) > 0$. Then*

$$K_n(a, b_1, b_2) \xrightarrow{p} 0. \tag{5.4}$$

Remark 5.1. We list the conditions which guarantee the convergence in (5.4).

1. From Theorem 5.3, it follows that if F is a large maxima concomitant distribution, $r_G = \infty$ and $\beta(a) > 0$, then (5.4) holds true.
2. From Theorem 5.2 it follows that if F is a c-stable maxima concomitant distribution (for some c, where $c \le r_G < \infty$) and $\beta(a) = 1$ then (5.4) holds true. We can say, alternatively here, that for all y

$$\lim_{x\to\infty} \frac{P_1(x, a, y, b_1, b_2)}{1 - H(x)} = 0.$$

In Theorem 5.4, conditions for strong convergence of $K_n(a, b_1, b_2)$ are found.

Theorem 5.4. *Let*

$$\int_{\mathbb{R}^2} \frac{P_1}{(1 - H(x))^2} F(\mathrm{d}x, \mathrm{d}y) < \infty. \tag{5.5}$$

Then $K_n(a, b_1, b_2) \overset{a.s.}{\to} 0$.

Proof. Indeed,

$$\begin{aligned}\sum_{n=2}^{\infty} P\{K_n(a, b_1, b_2) > 0\} &= \sum_{n=2}^{\infty}\left[1 - \int_{\mathbb{R}^2}\int (H(x) - P_1)^{n-1} F(\mathrm{d}x, \mathrm{d}y)\right] \\ &= \sum_{n=2}^{\infty}\int_{\mathbb{R}^2}\left[nH^{n-1}(x) - (H(x) - P_1)^{n-1}\right] F(\mathrm{d}x, \mathrm{d}y) \\ &= \int_{\mathbb{R}^2} \frac{P_1}{(1 - H(x))(1 - H(x) + P_1)} F(\mathrm{d}x, \mathrm{d}y) < \int_{\mathbb{R}^2} \frac{P_1}{(1 - H(x))^2} F(\mathrm{d}x, \mathrm{d}y).\end{aligned}$$

By Borel–Cantelli lemma, the result follows. □

Remark 5.2. Observe that Theorem 5.4 implies that if F is large maxima concomitant distribution, c-stable maxima concomitant distribution or even unstable maxima concomitant distribution and condition (5.5) holds, then $K_n(a, b_1, b_2) \overset{a.s.}{\to} 0$ ($\overset{p}{\to}$).

Remark 5.3. In the univariate case the condition for strong convergence of the number of near-maxima, the bivariate analogue of which is $K_n(a, \infty, \infty)$, is proposed by Li (1999). If

$$\int_{\mathbb{R}} \frac{H(x) - H(x - a)}{(1 - H(x))^2} \mathrm{d}F(x) < \infty, \tag{5.6}$$

then $K_n(a, \infty, \infty) \overset{a.s.}{\to} 0$. Observe that if Li's condition holds, then (5.5) also holds.

5.3. Examples

Example 5.1. Let

$$F(x, y) = \int_0^x \int_0^y \frac{1}{u} \mathrm{e}^{-u - v/u} \mathrm{d}v\mathrm{d}u \quad (x > 0, y > 0)$$

with marginal distribution $H(x) = 1 - \mathrm{e}^{-x}$ and density-distribution function $F(\mathrm{d}x, y) = \mathrm{e}^{-x}(1 - \mathrm{e}^{-y/x})\mathrm{d}x$. For small $\varepsilon > 0$ and fixed y choose x_0 such that $1 - \mathrm{e}^{-y/x} < \varepsilon$ $(x > x_0)$. Then

$$\begin{aligned}P\{Y_{\max}^{(n)} \le y\} &= o(1) + n\int_{x_0}^{\infty} H^{n-1}(x) F(\mathrm{d}x, y) \\ &= o(1) + n\int_{x_0}^{\infty} (1 - \mathrm{e}^{-x})^{n-1}\mathrm{e}^{-x}(1 - \mathrm{e}^{-y/x})\mathrm{d}x < o(1) + \varepsilon.\end{aligned}$$

This means that F is a large maxima concomitant distribution and $\overline{M}_n \overset{p}{\to} (\infty, \infty)$. The limit in (5.2) exists and $\beta(a) > 0$. By Theorem 5.3, $K_n(a, b_1, b_2) \overset{p}{\to} 0$. However,

$$\frac{P_1}{(1 - H(x))^2} f(x, y) \sim \frac{(\mathrm{e}^a - 1)\mathrm{e}^{2k}}{x} \quad (x \to \infty, y = kx),$$

where $f(x, y)$ is the joint density. This indicates that condition (5.5) does not hold and Theorem 5.4 cannot be applied here.

Example 5.2. Let

$$F(x,y)=1-\mathrm{e}^{-x}-\frac{1-\mathrm{e}^{-x(y+1)}}{y+1}\quad (x>0,y>0)$$

with marginal distributions $H(x)=1-\mathrm{e}^{-x}$ $(x>0)$ and $G(y)=1-\frac{1}{y+1}$ $(y>0)$. Condition (2.2) holds here with $c=0$, i.e. F is a 0-stable maxima concomitant distribution and $\overline{M}_n \xrightarrow{p} (\infty,0)$. The limits in (5.2) and (5.3) exist and $\beta(a)=\mathrm{e}^{-a}$. By Theorem 5.2 we obtain the convergence in distribution $K_n(a,0,b_2)\xrightarrow{d} Geo(\mathrm{e}^{-a})$.

Example 5.3. Let

$$F(x,y)=1-\mathrm{e}^{-y}-\frac{1-\mathrm{e}^{-y(x+1)}}{x+1}\quad (x>0,y>0)$$

with marginal distributions

$$G(y)=1-\mathrm{e}^{-y}\ (y>0)\quad\text{and}\quad H(x)=1-\frac{1}{x+1}\ (x>0).$$

Condition (2.2) holds here with $c=0$, i.e. F is a 0-stable maxima concomitant distribution and $\overline{M}_n \xrightarrow{p} (\infty,0)$. The limits in (5.2) and (5.3) exist and $\beta(a)=1$. By Theorem 5.2, $K_n(a,0,b_2)\xrightarrow{p} 0$. Condition (5.6) holds true, and by Remark 5.3, $K_n(a,0,b_2)\xrightarrow{a.s.} 0$.

Acknowledgement

We are grateful to the referee for the helpful and interesting suggestions.

Appendix

Proof of Proposition 5.1. We present the proof for the case when $c<r_G$. If $c=r_G$ the proof utilizes the same argument.

Let us recall that for a c-stable maxima concomitant distribution F

$$n\int_{\mathbb{R}} H^{n-1}(x)F(\mathrm{d}x,y)\to 1\quad (y>c) \tag{A.1}$$

and

$$n\int_{\mathbb{R}} H^{n-1}(x)F(\mathrm{d}x,y)\to 0\quad (y<c). \tag{A.2}$$

Observe that for any continuous F,

$$n\int_{\mathbb{R}} F(x,y)H^{n-1}(x)\mathrm{d}F(x)\to G(y). \tag{A.3}$$

Define continuous function δ by

$$\delta(x,y)=G(y)-F(x,y).$$

The function δ is decreasing in x and for any fixed y

$$\delta(-\infty,y)=G(y),\quad \delta(\infty,y)=0.$$

It follows from (A.1)–(A.3) that if F is a c-stable maxima concomitant distribution, then

$$n(n-1)\int_{\mathbb{R}}\delta(x,y)H^{n-2}(x)\mathrm{d}F(x)\to 1\quad (y>c,n\to\infty) \tag{A.4}$$

and

$$n(n-1)\int_{\mathbb{R}}\delta(x,y)H^{n-2}(x)\mathrm{d}F(x)\to 0\quad (y<c,n\to\infty). \tag{A.5}$$

The convergency in (A.4) and (A.5) allow us to compare the functions δ and $1-H(x)$ at infinity. If F is a c-stable maxima concomitant distribution, then by the existence of the limit in (5.3),

$$\frac{\delta(x,y)}{1-H(x)}\to 1\quad (x\to\infty,\ y>c)$$

and

$$\frac{\delta(x,y)}{1-H(x)} \to 0 \quad (x \to \infty,\ y < c).$$

Finally, we get

$$\begin{aligned}\frac{P_1(x,a,y,b_1,b_2)}{1-H(x)} &= \frac{\delta(x-a,y+b_2)}{1-H(x-a)} \cdot \frac{1-H(x-a)}{1-H(x)} - \frac{\delta(x,y+b_2)}{1-H(x)} + \frac{\delta(x,y-b_1)}{1-H(x)} \\ &\quad - \frac{\delta(x-a,y-b_1)}{1-H(x-a)} \cdot \frac{1-H(x-a)}{1-H(x)} \\ &\to 1 \cdot \frac{1}{\beta(a)} - 1 + 0 - 0 \cdot \frac{1}{\beta(a)}. \quad \square\end{aligned}$$

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