Stat Papers (2010) 51:227–240
DOI 10.1007/s00362-008-0134-3

REGULAR ARTICLE

Limit results for ordered uniform spacings

Ismihan Bairamov · Alexandre Berred · Alexei Stepanov

Received: 30 August 2007 / Accepted: 5 March 2008 / Published online: 10 April 2008

Abstract Let $\Delta_{k:n} = X_{k,n} - X_{k-1,n} (k = 1, 2, \ldots, n+1)$ be the spacings based on uniform order statistics, provided $X_{0,n} = 0$ and $X_{n+1,n} = 1$. Obtained from uniform spacings, ordered uniform spacings $0 = \Delta_{0,n} < \Delta_{1,n} < \cdots < \Delta_{n+1,n}$, are discussed in the present paper. Distributional and limit results for them are in the focus of our attention.

Keywords Uniform distribution · Order statistics · Spacings · Ordered spacings · Limit theorems

Mathematics Subject Classification (2000) 60G70 · 62G30

1 Introduction

Suppose in the following, $X_1, \ldots, X_n$ are independent and identically distributed random variables having continuous F with support $[a, b]$. Let $a = X_{0,n} \leq X_{1,n} \leq$

I. Bairamov
Department of Mathematics, Izmir University of Economics,
35330 Balcova, Izmir, Turkey
e-mail: ismihan.bayramoglu@ieu.edu.tr

A. Berred
Faculté des Sciences et Techniques, Université du Havre,
B. P. 540,76058 Le Havre Cedex, France
e-mail: berred@univ-lehavre.fr

A. Stepanov (✉)
Department of Mathematics, Kaliningrad State Technical University,
Sovietsky Prospect 1, Kaliningrad 236000, Russia
e-mail: alexei@step.koenig.ru

$\cdots \leq X_{n,n} \leq X_{n+1,n} = b$ and $\Delta_{k:n} = X_{k,n} - X_{k-1,n}$ $(k = 1, \ldots, n+1)$ be the order statistics and the spacings based on this sample.

Spacings play an important role in many research areas such as goodness of fit tests, reliability analysis, survival analysis and applications to the statistical theory. For more information on spacings we refer to Pyke (1965); Hall (1984); Arnold et al. (1992); Kotz and Nadarajah (2000); Nevzorov (2001), and David and Nagaraja (2003).

The asymptotic theory of spacings has been developed intensively. We mention several results from the asymptotic theory of uniform spacings. Levy (1939) obtained the asymptotic distribution of the maximal spacing $\Delta_{n+1,n} = \max_{1 \leq i \leq n+1} \Delta_{i:n}$ from the uniform on $[-1, 1]$ distribution and showed that

$$P\left\{\frac{n\Delta_{n+1,n}}{\log n} \leq x\right\} \to \exp(-\exp(-x)) \qquad (x \in R).$$

Devroye (1981) established that

$$\lim_{n\to\infty} \sup \left[\frac{n\Delta_{n+1,n} - \log n}{2\log\log n}\right] = 1 \qquad \text{a.s.}$$

In the case of continuous F, Kimball (1947) studied the asymptotic normality of the statistic

$$\alpha_n = \sum_{i=1}^{n+1}\left(F(X_{i,n}) - F(X_{i-1,n}) - \frac{1}{n+1}\right)^2 \qquad (X_{0,n} = -\infty,\ X_{n+1,n} = \infty),$$

which is the measure of the deviation from uniform spacing. Moran (1947) considered a similar statistic

$$\beta_n = \sum_{i=1}^{n+1}\left(F(X_{i,n}) - F(X_{i-1,n})\right)^2$$

and proved that β_n is asymptotically normal. Sherman (1950) has studied another deviation from uniform spacing

$$\varpi_n = \sum_{i=1}^{n+1}\left|F(X_{i,n}) - F(X_{i-1,n}) - \frac{1}{n+1}\right|$$

and showed that the distribution of the standardized variable

$$\frac{\varpi_n - E(\varpi_n)}{Var(\varpi_n)}$$

approaches to the Normal law. These results have been used for constructing the goodness of fit tests. One of such recent tests was proposed by Bairamov and Ozkaya

(2000). Let $Y_1, \ldots, Y_m$ be another sample of independent observations with distribution G concentrated on the same support. Let $\mu_m(n, i, j)$ be the number of those $Y_1, \ldots, Y_m$ that belong to the interval $(X_{i,n}, X_{j,n})$, $1 \le i < j \le n$ and $p_k = P\left\{Y_j \in (X_{k-1,n}, X_{k,n})\right\}$ $(i = 1, \ldots, m)$. Under the hypothesis $H_0 : F = G$, it is true that $p_k = 1/(n+1)$. Bairamov and Ozkaya (2000) derived the exact distribution of the test statistic

$$\eta_{n,m} = \sum_{k=1}^{n+1} \left[\frac{\mu_m(n, k-1, k)}{m} - p_k \right]^2$$

under the hypothesis H_0 and constructed a consistent test for testing H_0 versus general classes of alternatives. Hu and Zhuang (2005) have investigated stochastic comparisons of general p-spacings for a subclass of generalized order statistics in the likelihood ratio and hazard rate orders. Recently, Eryilmaz and Stepanov (2008) studied runs based on uniform order statistics and derived some limit results for uniform spacings from the results obtained for runs.

Relatively little was done for investigating ordered uniform spacings. We mention here the papers of Levy (1939); Renyi (1953); Barton and David (1956); Weiss (1959, 1969); Pyke (1965); Abramson (1966) and Devroye (1981), where distributional and asymptotic results were obtained basically for the smallest and largest uniform spacings. In our work, we present a general approach for studying these matters and derive new limit results for ordered uniform spacings.

The rest of this paper is organized as follows. In Sect. 2, distributional results for ordered uniform spacings are given. The asymptotic behavior of expectations of such spacings are considered in Sect. 3. In Sects. 4, we study asymptotic properties of normalized ordered uniform spacings. Finally, in Sect. 5, some applications for exceedance statistics associated with ordered uniform spacings are presented.

Further in the paper, with exception of Sect. 5, we suppose that $F(x) = x$ and $a = 0, b = 1$.

2 Exact distributions of ordered uniform spacings

The distributions of the uniform spacings (see, for example, Feller 1967) are given as

$$\begin{aligned}
&P\{\Delta_{k:n} \le x\} = 1 - (1-x)^n \qquad (0 \le x \le 1, 1 \le k \le n+1),\\
&P\left\{\Delta_{1:n} > x_1, \ldots, \Delta_{r:n} > x_r\right\}\\
&\quad = (1 - x_1 - \cdots - x_r)^n \quad \left(\sum_{i=1}^{r} x_i \le 1, 1 \le r \le n+1 \right).
\end{aligned}$$

It immediately follows that uniform spacings are exchangeable random variables. Arranging the spacings $\Delta_{1:n}, \ldots, \Delta_{n+1:n}$ in increasing order we obtain ordered uniform spacings

$$0 = \Delta_{0,n} < \Delta_{1,n} < \cdots < \Delta_{n+1,n}.$$

Several representations are known for such spacings. One of them is given in Pyke (1965)

$$(n-k+1)(\Delta_{k+1,n}-\Delta_{k,n}) \stackrel{d}{=} \Delta_{1:n} \qquad (k=0,\ldots n). \tag{1}$$

Suppose in the following, ν_i $(i \geq 1)$ are independent unit exponential variables. Renyi (1953) found the connection between uniform ordered spacings and the variables ν_i:

$$\Delta_{k,n} \stackrel{d}{=} \frac{\nu_1/(n+1)+\nu_2/n+\cdots+\nu_k/(n-k+2)}{\nu_1+\cdots+\nu_{n+1}} \quad (k=1,\ldots,n+1). \tag{2}$$

The joint distribution of ordered uniform spacings is known long ago, see for example, Exercise 5.4.3 of David and Nagaraja (2003). However, we do not know if, somewhere, the individual distributions of $\Delta_{k,n}$ were obtained for all $k=1,\ldots,n+1$. In Lemma 1, we give such distributions.

Let the interval [0, 1) be presented as the sum of non-overlapping intervals

$$[0,1) = I_{1,n} \bigcup I_{2,n} \bigcup \cdots \bigcup I_{n+1,n},$$

where $I_{1,n} = \left[0, \frac{1}{n+1}\right)$ and $I_{m,n} = \left[\frac{1}{n+3-m}, \frac{1}{n+2-m}\right)$ $(m=2,\ldots,n+1)$.

Lemma 1 *The distribution of the k-th* $(1 \leq k \leq n+1)$ *ordered uniform spacing is presented by*

$$\begin{aligned} &P\{\Delta_{k,n} \leq x\} = 0 \qquad (x \leq 0),\\ &P\{\Delta_{k,n} > x\} = 0 \qquad (x(n+2-k) \geq 1), \end{aligned} \tag{3}$$

and for $x \in I_{m,n}$, $m=1,\ldots,k$

$$P\{\Delta_{k,n} > x\} = (-1)^{k-1}(n+1)\binom{n}{k-1}\sum_{i=m}^{k}\frac{(-1)^{i-1}}{(n+2-i)}\binom{k-1}{i-1}(1-x(n+2-i))^n. \tag{4}$$

Let us consider (4) for certain values of k. The distribution of the minimal spacing $(k=1)$ is given by

$$P\{\Delta_{1,n} > x\} = (1-x(n+1))^n \quad x \in I_{1,n}. \tag{5}$$

For $k=2$ we have

$$P\{\Delta_{2,n} > x\} = \begin{cases} (n+1)(1-nx)^n - n(1-(n+1)x)^n, & x \in I_{1,n} \\ (n+1)(1-nx)^n, & x \in I_{2,n} \end{cases},$$

The distribution of the maximal spacing is the following

$$P\{\Delta_{n+1,n} > x\} = (-1)^n(n+1)\sum_{i=m}^{n+1}\frac{(-1)^{i-1}}{(n+2-i)}\binom{n}{i-1}(1-x(n+2-i))^n, \tag{6}$$

where $x \in I_{m,n}$, $m = 1, \ldots, n+1$. It follows from (6) that $P\{\Delta_{n+1,n} > \frac{1}{n+1}\} = 1$. This means that the value of the maximal spacing cannot be less than $1/(n+1)$.

Proof There are two possible ways of obtaining (4). The first way, and surprisingly not the easiest one, supposes using the joint distribution of $\Delta_{1,n}, \ldots, \Delta_{k,n}$, given among others, in David and Nagaraja (2003). The second way is based on attracting (2). We choose the second way.

The following auxiliary lemma is used for proving Lemma 1.

Lemma 2 *Let $\nu_1, \ldots, \nu_k$ be independent unit exponential variables. Suppose for some $m = 1, \ldots, k$ the numbers $s_1, \ldots, s_{m-1}$ are negative and the numbers $s_m, \ldots, s_k$ are positive. Let also all the s_i $(i = 1, \ldots, k)$ are different. Then*

$$P\{s_1\nu_k + \cdots + s_k\nu_k > x\} = (-1)^{k-1} \sum_{i=m}^{k} e^{-\frac{x}{s_i}} s_i^{k-1} \prod_{j=1, j\neq i}^{k} \frac{1}{s_j - s_i} \qquad (x > 0),$$

where $\prod_{j=1,j\neq i}^{1} \frac{1}{s_j - s_i} = 1$.

Let us come back to the proof of Lemma 1. Equality (2) entails

$$P\{\Delta_{k,n} > x\} = P\{\nu_{k+1} + \cdots + \nu_{n+1} < s_1\nu_1 + \cdots + s_k\nu_k\}, \quad x \in I_{m,n+1},$$

where $s_i = \frac{1}{x(n+2-i)} - 1$. Observe that $s_1, \ldots, s_{m-1}$ are negative, $s_m, \ldots, s_k$ are positive and all the s_i $(i = 1, \ldots, k)$ are different. Because the variable $\nu_{k+1} + \cdots + \nu_{n+1}$ has the gamma distribution with parameters $(n - k + 1, 1)$, we have

$$P\{\Delta_{k,n} > x\} = \frac{1}{(n-k)!} \int_0^\infty e^{-z} z^{n-k} P\{s_1\nu_1 + \cdots + s_k\nu_k > z\}\, dz. \tag{7}$$

Now, equality (4) can be obtained from (7) by means of Lemma 2. □

3 Expectations of ordered uniform spacings

The expectation of k-th ordered uniform spacing

$$E\Delta_{k,n} = \frac{1}{n+1} \sum_{i=n+2-k}^{n+1} \frac{1}{i} \quad (k = 1, \ldots, n+1)$$

can be obtained either directly from (4), or from the recurrence expression

$$E\Delta_{k+1,n} = E\Delta_{k,n} + \frac{1}{(n+1)(n+1-k)} \quad (k = 0, \ldots, n),$$

which, in its turn, follows from (1).

Analyzing the asymptotic behavior of the expectation of $\Delta_{k,n}$, we concentrate on five basic cases.

1. For finite $k \geq 1$ and $n \to \infty$ we have

$$E\Delta_{k,n} = O\left(\frac{1}{n^2}\right) \to 0, \tag{8}$$

2. If $k = k_n \to \infty$ such that $k_n = o(n)$, then

$$E\Delta_{k_n,n} \sim \frac{k_n}{n^2} \to 0. \tag{9}$$

3. If $k_n = \alpha n$ $(0 < \alpha < 1, n \to \infty)$, then

$$E\Delta_{k_n,n} \sim \frac{-\log(1-\alpha)}{n} \to 0. \tag{10}$$

4. If $k = k_n \to \infty$ such that $k_n = o(n)$, then

$$E\Delta_{n+2-k_n,n} \sim \frac{\log(n/o(n))}{n} \to 0. \tag{11}$$

5. For finite $k \geq 1$ and $n \to \infty$ we have

$$E\Delta_{n+2-k,n} \sim \frac{\log n}{n} \to 0. \tag{12}$$

These five cases give us further orientation in obtaining limit laws for ordered uniform spacings.

4 Limit results for ordered uniform spacings

In this section, we analyze the asymptotic behavior of $\Delta_{k,n}$ with different normalizing constants. Limit properties of $\Delta_{k,n}$ and $\Delta_{n-k+2,n}$ ($k \geq 1$ is fixed and $n \to \infty$) will be considered in Subsects. 4.1, 4.2, respectively. Asymptotic results for $\Delta_{k_n,n}$, when k_n grows as n grows, will be presented in Subsect. 4.3.

4.1 Limit behavior of $\Delta_{k,n}$

In this subsection we analyze the limit behavior of $\Delta_{k,n}$ when $k \geq 1$ is fixed and $n \to \infty$.

Since $\frac{1}{n+2-k} \to 0$, the right end of the support of $\Delta_{k,n}$ tends to zero and $\Delta_{k,n} \to 0$ $a.s.$ for any fixed $k \geq 1$. This observation is, of course, trivial. More complicated forms of limit laws for k-th ordered uniform spacing when k is fixed are obtained below.

Observe that it follows from (8) and Chebyshov's inequality that $a_n \Delta_{k,n} \to_p 0$ as $a_n = o(n^2)$. Making use of (4), we find in Theorem 1 the conditions for $a_n \Delta_{k,n} \to 0\ a.s.$

Theorem 1 *Let a_n be a growing sequence such that*

$$a_n = o(n^2) \qquad (n \to \infty) \tag{13}$$

and for some fixed $k \geq 1$

$$\sum_{n=1}^{\infty} n^{k-1} e^{-\frac{n^2}{a_n}} < \infty. \tag{14}$$

Then for $i \leq k$

$$a_n \Delta_{i,n} \to 0 \qquad a.s. \tag{15}$$

Proof Obviously, it is enough to prove this result only for the sequences a_n that satisfy (13), (14) and the condition $\frac{a_n}{n} \to \infty$.

The support of $a_n \Delta_{k,n}$ $(n \geq 1)$ is presented as the sum of non-overlapping intervals

$$\widetilde{I}_{1,n} \bigcup \cdots \bigcup \widetilde{I}_{k,n},$$

where

$$\widetilde{I}_{1,n} = \left[0, \frac{a_n}{n+1}\right), \quad \widetilde{I}_{m,n} = \left[\frac{a_n}{n+3-m}, \frac{a_n}{n+2-m}\right) \quad (m = 2, \ldots, k).$$

If the condition $\frac{a_n}{n} \to \infty$ holds then for any small fixed $x > 0$ an n_0 exists such that $x \in \widetilde{I}_{1,n}$ $(n > n_0)$. Then

$$\begin{aligned}
\sum_{n=k-1}^{\infty} P\{a_n \Delta_{k,n} > x\} &= \sum_{n=k-1}^{n_0} P\{a_n \Delta_{k,n} > x\} \\
&\quad + \sum_{n=n_0+1}^{\infty} (-1)^{k-1}(n+1)\binom{n}{k-1} \sum_{i=1}^{k} \frac{(-1)^{i-1}}{(n+2-i)} \binom{k-1}{i-1} \left(1 - \frac{x(n+2-i)}{a_n}\right)^n \\
&< c_{k,n_0} + \sum_{n=n_0+1}^{\infty} \frac{(n+1)^k}{n+2-k} \sum_{i=1}^{k} \left(1 - \frac{x(n+2-i)}{a_n}\right)^n \\
&= c_{k,n_0} + \sum_{n=n_0+1}^{\infty} O\left(n^{k-1} e^{-\frac{n^2}{a_n}}\right) < \infty,
\end{aligned}$$

where $c_{k,n_0} = \sum_{n=k-1}^{n_0} P\{a_n \Delta_{k,n} > x\}$. It follows from Borel–Cantelli lemma that $a_n \Delta_{k,n} \to 0\ a.s.$, which, in its turn, implies (15). □

If $a_n = O(n^2)$ we have another type of convergence.

Theorem 2 *For any fixed $k \geq 1$ and $n \to \infty$*

$$P\{n^2 \Delta_{k,n} > x\} \to \frac{1}{(k-1)!} \int_x^\infty e^{-u} u^{k-1} du \qquad (x > 0). \tag{16}$$

Proof Observe that the support of $n^2\Delta_{k,n}(n \to \infty)$ coincides with the interval $(0, \infty)$. Making use of (2), we get

$$P\{n^2 \Delta_{k,n} > x\} = P\left\{\left(\frac{n}{n+1} - \frac{x}{n+1}\right)\nu_1 + \cdots + \left(\frac{n}{n+2-k} - \frac{x}{n+1}\right)\nu_k \right.$$
$$\left. > x\frac{\nu_{k+1} + \cdots + \nu_{n+1}}{n}\right\}.$$

Observe that $\frac{\nu_{k+1}+\cdots+\nu_{n+1}}{n} \to 1$ $a.s.$ Then

$$\lim_{n\to\infty} P\{n^2 \Delta_{k,n} > x\} = P\{\nu_1 + \cdots + \nu_k > x\}, \tag{17}$$

which implies (16). $\square$

Theorem 3 *Let a_n be a growing sequence such that $\sum_{n=1}^\infty \frac{n^2}{a_n} < \infty$. Then for any $k \geq 1$*

$$a_n \Delta_{k,n} \to \infty \qquad a.s.$$

Proof Obviously, it is enough to prove the theorem for $k = 1$. Observe that the support of $a_n\Delta_{1,n+1}(n \to \infty)$ coincides with the interval $(0, \infty)$. We have for any $x > 0$

$$\sum_{n=1}^\infty P\{a_n \Delta_{1,n} < x\} = \sum_{n=1}^\infty \left(1 - \left(1 - \frac{x(n+1)}{a_n}\right)^n\right).$$

Under the condition $\frac{n^2}{a_n} \to 0$ the expression $1 - \left(1 - \frac{x(n+1)}{a_n}\right)^n$ for large n behaves like $x\frac{n^2}{a_n}$. The series given in Theorem 3 converges, and the result follows from Borel–Cantelli lemma. $\square$

From the proof of Theorem 3 one can get that $a_n\Delta_{k,n} \to_p \infty$, if $\frac{n^2}{a_n} \to 0$.

Corollary 1 *As a consequence of Theorem 3, we have*

$$n^{3+\delta}\Delta_{k,n} \to \infty \qquad a.s. \qquad (\delta > 0).$$

4.2 Limit behavior of $\Delta_{n-k+2,n}$

In this subsection we analyze the limit behavior of $\Delta_{n-k+2,n}$ when $k \geq 1$ is fixed and $n \to \infty$.

It follows from (12) and Chebyshov's inequality that $a_n \Delta_{n-k+2,n} \to_p 0$ if $a_n = o\left(\frac{n}{\log n}\right)$. In particular, we have $n^\alpha \Delta_{n-k+2,n} \to_p 0$ $(0 \leq \alpha < 1)$. When $\alpha = 0$ a strong limit result holds for $\Delta_{n-k+2,n}$.

Theorem 4 *For $k \geq 1$ and $n \to \infty$ the following asymptotic property holds*

$$\Delta_{n-k+2,n} \to 0 \qquad a.s. \tag{18}$$

Proof Obviously, it is enough to prove this theorem for $k = 1$.

We will make use of (1). We have

$$P\{\Delta_{n+1,n} > x\} = P\left\{\frac{\nu_1/(n+1) + \nu_2/n + \cdots + \nu_{n+1}}{\nu_1 + \nu_2 + \cdots + \nu_{n+1}} > x\right\} = P_1.$$

For any small fixed $x \in (0, 1)$ a fixed number n_0 exists such that $\frac{1}{n_0+1} < x \leq \frac{1}{n_0}$. Then for $n > n_0$

$$P_1 = P\left\{(1-x)\nu_{n+1} + \cdots + \left(\frac{1}{n_0} - x\right)\nu_{n-n_0+2} \right.$$
$$\left. > \left(x - \frac{1}{n_0+1}\right)\nu_{n-n_0+1} + \cdots + \left(x - \frac{1}{n+1}\right)\nu_1\right\}.$$

Observe that $1 - x > \cdots > \frac{1}{n_0} - x \geq 0$ and $0 < x - \frac{1}{n_0+1} < \cdots < x - \frac{1}{n+1}$. We have

$$P_1 < P_2 = P\left\{\left[\nu_{n+1} + \cdots + \nu_{n-n_0+2}\right] > x_0\left[\nu_{n-n_0+1} + \cdots + \nu_1\right]\right\},$$

where $x_0 = \frac{1}{1-x}\left(x - \frac{1}{n_0+1}\right) > 0$. Then

$$\sum_{n=n_0}^{\infty} P\{\Delta_{n+1,n} > x\} < \sum_{n=n_0}^{\infty} P_2$$
$$= \sum_{n=n_0}^{\infty} \frac{1}{(n_0-2)!}\int_R e^{-z} z^{n_0-2} P\{\nu_1 + \cdots + \nu_{n-n_0+1} < x_0 z\} = x_0(n_0-1) < \infty.$$

The result readily follows from Borel–Cantelli lemma. □

Theorem 5 *Let a_n be a growing sequence such that $\sum_{n=1}^{\infty} \frac{n}{a_n} < \infty$. Then for $k \geq 1$*

$$a_n \Delta_{n-k+2,n} \to \infty \qquad a.s. \tag{19}$$

Proof We prove this theorem for $k = 1$. Making use of (2), we have

$$\sum_{n=1}^{\infty} P\{a_n(\Delta_{n+1,n} - \Delta_{n,n}) \le x\} = \sum_{n=1}^{\infty} P\{a_n \Delta_{1:n} \le x\} = \sum_{n=1}^{\infty} \left(1 - \left(1 - \frac{x}{a_n}\right)^n\right).$$

In the case when $\frac{n}{a_n} \to 0$, the expression $1 - \left(1 - \frac{x}{a_n}\right)^n$ for large n behaves like $\frac{xn}{a_n}$, and the series given in Theorem 5 converges. From Borel–Cantelli lemma we can obtain that $a_n(\Delta_{n+1,n} - \Delta_{n,n}) \to \infty$ $a.s.$, and the result follows. □

From the proof of Theorem 5 we get that $a_n \Delta_{n-k+2,n} \to_p \infty$, if $\frac{n}{a_n} \to 0$.

Corollary 2 *As a consequence of Theorem 5, we have*

$$n^{2+\delta} \Delta_{n-k+2,n} \to \infty \qquad a.s. \qquad (\delta > 0).$$

4.3 Limit behavior of $\Delta_{k_n,n}$

(i) Let $k_n \to \infty$ and $k_n = o(n)$.

We have from (9) and Chebyshov's inequality that $a_n \Delta_{k_n,n} \to_p 0$ if $a_n = o\left(\frac{n^2}{k_n}\right)$. However, if a_n grows a little faster, we have an inverse result.

Theorem 6 *Let $k_n \to \infty$ and $k_n = o(n)$ as $n \to \infty$. Then $n^2 \Delta_{k_n,n} \to_p \infty$.*

Proof From (17), we get

$$\lim_{n\to\infty} P\{n^2 \Delta_{k_n,n} \le x\} = \lim_{k_n\to\infty} P\{\nu_1 + \cdots + \nu_{k_n} \le x\} = 0 \qquad (x > 0).$$

The result follows. □

Observe that under the conditions of Theorem 3 for case (i) we also have $a_n \Delta_{k_n,n} \to \infty$ $a.s.$

(ii) Let $k_n = \alpha n$ $(0 < \alpha < 1, n \to \infty)$.

From (10) and Chebyshov's inequality it follows that $a_n \Delta_{k_n,n} \to_p 0$, if $a_n = o(n)$. Having the same type of growth of a_n, we are able to prove a stronger limit result.

Theorem 7 *Let $k_n = \alpha n$ $(0 < \alpha < 1)$ and $a_n = o(n)$. Then*

$$a_n \Delta_{k_n,n} \to 0, \qquad a.s. \tag{20}$$

Proof Let us make use of (1). Choose some fixed small $x > 0$ and large n_0. Then

$$\sum_{n=n_0}^{\infty} P\{a_n \Delta_{k_n,n} > x\} = \sum_{n=n_0}^{\infty} P\{A_{n,x}\},$$

where

$$A_{n,x} = \left\{ \left(\frac{a_n}{x(n+1)} - 1 \right) \nu_1 + \cdots + \left(\frac{a_n}{x(n-k_n+2)} - 1 \right) \nu_{k_n} > \nu_{k_n+1} + \cdots + \nu_{n+1} \right\}.$$

Since some n all the coefficients $\left(\frac{a_n}{x(n+1)} - 1\right) \nu_1, \ldots, \left(\frac{a_n}{x(n-k_n+2)} - 1\right)$ in $A_{n,x}$ will be negative. The result follows from Borel–Cantelli lemma. □

Considering the convergence of $\Delta_{k_n,n}$ to infinity, in the present work we are able only to restate the same limit laws that were already established in (i).

(iii) Let us discuss the behavior of $\Delta_{n-k_n+2,n}$, when $k_n = o(n)$. Analyzing the method of the proof of Theorem 5, we formulate the following. Let $\frac{k_n n}{a_n} \to 0$, then $a_n \Delta_{n-k_n+2,n} \to_p \infty$. If $\sum_{n=1}^{\infty} \frac{k_n n}{a_n} < \infty$, then $a_n \Delta_{n-k_n+2,n} \to \infty \; a.s.$

(iv) At the end of this section we present a result that has a more general character. With a slight modification in the proof of Theorem 7 one can obtain Theorem 8.

Theorem 8 *Let $k_n \to \infty$, $k_n \le n+1$ and $a_n = o(k_n)$. Then*

$$a_n \Delta_{n+2-k_n,n+1} \to 0, \qquad a.s. \tag{21}$$

If follows from Theorem 8 that if $a_n = o(n)$ then $a_n \Delta_{k_n,n} \to 0 \; a.s.$

5 Statistics based on ordered spacings

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be independent and identically distributed independent samples with continuous distributions F and G, respectively. Let the common support $[a, b]$ of these samples be bounded. As before, the designations $\Delta_{i:n}$ and $\Delta_{k,n}$, respectively, mean the i-th spacing and the k-th ordered spacings based on the fist sample. Let θ_k be the index of the k-th ordered spacing, i.e.,

$$\theta_k = j \Leftrightarrow X_{j,n} - X_{j-1,n} = \Delta_{k,n}, \qquad (j = 1, \ldots, n+1).$$

Define an exceedance statistic $\mu_k(m)$ by

$$\mu_k(m) = \#\{Y_i : Y_i \in (X_{\theta_k - 1:n}, X_{\theta_k:n})\} \qquad (1 \le i \le m).$$

The variable $\mu_k(m)$ counts the number of Y_i that fall in the random interval $(X_{\theta_k-1,n}, X_{\theta_k,n})$.

Theorem 9 presented below is useful for analyzing the asymptotic behavior of the exceedance statistic $\mu_k(m)$.

Theorem 9 *Let $X_1, \ldots, X_m$ be independent variables with distribution F. Assume that Z_1 and Z_2 are independent of X_i continuous random variables with joint distribution function $F_{Z_1,Z_2}(t,s)$, such that $Z_1 \le Z_2$ $a.s.$ Let X, Z_1 and Z_2 have common support. Denote*

$$\widetilde{\mu}(m) = \#\{X_i : X_i \in (Z_1, Z_2)\}.$$

Then

$$\lim_{m\to\infty}\sup\left|P\left\{\frac{\widetilde{\mu}(m)}{m}\le x\right\}-P\{F(Z_2)-F(Z_1)\le x\}\right|=0.$$

Proof Define

$$\tau_i=\begin{cases}1, & \text{if } X_i\in(Z_1,Z_2)\\ 0, & \text{if } X_i\notin(Z_1,Z_2)\end{cases},\quad i=1,\dots,m.$$

Then

$$\widetilde{\mu}(m)=\sum_{i=1}^{m}\tau_i=\sum_{i=1}^{m}I_{\{(Z_1,Z_2)\}}(X_i),\tag{22}$$

where

$$I_A(x)=\begin{cases}1, & \text{if } x\in A\\ 0, & \text{if } x\notin A\end{cases}.$$

By (22)

$$\begin{aligned}P\left\{\frac{\widetilde{\mu}(m)}{m}\le x\right\}&=P\left\{\frac{1}{m}\sum_{i=1}^{m}I_{\{(Z_1,Z_2)\}}(X_i)\le x\right\}\\ &=P\left\{\int_{-\infty}^{\infty}I_{\{(Z_1,Z_2)\}}(u)dF_m^*(u)\le x\right\},\end{aligned}$$

where $F_m^*(u)$ denotes the empirical distribution function of the sample $X_1,\dots,X_m$. We consider the sample $X_1,\dots,X_m$ as an independent copies of the random variable X defined in probability space (Ω,F,P), where Ω is a set of points, F is a σ-field of subsets of Ω, $F(x)=P\{\omega:X(\omega)\le x\}$. Denote

$$G^*(F)=\int_{-\infty}^{\infty}I_{\{(t,s)\}}(u)dF(u)\qquad(t,s\in R,t\le s)$$

$$G(F)=\int_{-\infty}^{\infty}I_{\{(Z_1,Z_2)\}}(u)dF(u),$$

where $G(f)=G(f)(\omega)$ is a random variable defined in probability space $\{\Omega,F,P\}$. One can write

$$P\left\{\frac{\nu(m)}{m}\le x\right\}=P\left\{\int_{-\infty}^{\infty}I_{\{(Z_1,Z_2)\}}(u)dF_m^*(u)\le x\right\}=P\left\{G(F_m^*)\le x\right\}.$$

Using the continuity with respect to uniform metric and following Glivenko–Cantelli theorem, we get

$$
\begin{aligned}
&P\{\omega : \lim_{m\to\infty} G(F_m^*) = G(F)\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\left\{\lim_{m\to\infty}\int_{-\infty}^{\infty} I_{\{(Z_1,Z_2)\}}(u)dF_m^*(u)\right. \\
&= \left.\int_{-\infty}^{\infty} I_{\{(Z_1,Z_2)\}}(u)dF(u) \mid Z_1 = t, Z_2 = s\right\} dF_{Z_1,Z_2}(t,s) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\left\{\lim_{m\to\infty}\int_{-\infty}^{\infty} I_{\{(t,s)\}}(u)dF_m^*(u) = \int_{-\infty}^{\infty} I_{\{(t,s)\}}(u)dF(u)\right\} dF_{Z_1,Z_2}(t,s) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\left\{\lim_{m\to\infty} G^*(F_m^*) = G^*(F)\right\} dF_{Z_1,Z_2}(t,s) = 1.
\end{aligned}
$$

Therefore $G(F_m^*) \to G(F)$ almost sure in $\{\Omega, F, P\}$, which implies convergence in distribution. Then

$$
P\{G(F) \le x\} = P\left\{\int_{-\infty}^{\infty} I_{\{(Z_1,Z_2)\}}(u)dF(u) \le x\right\} = P\{F(Z_2) - F(Z_1) \le x\}.
$$

□

Theorem 10 is a simple consequence of Theorem 9.

Theorem 10 *The asymptotic distribution of $\mu_k(m)/m$ is presented by*

$$
\lim_{m\to\infty}\sup_{0\le x<1} \mid P\left\{\frac{\mu_k(m)}{m} \le x\right\} - P\{F(X_{\theta_k:n}) - F(X_{\theta_k-1:n}) \le x\} \mid= 0.
$$

Note that for the uniform distribution $F(X_{\theta_k,n}) - F(X_{\theta_k-1,n}) = \Delta_{k,n}$. From Theorem 10 we obtain the following.

Corollary 3 *Let $X_1,\dots,X_n$ and $Y_1,\dots,Y_m$ be independent samples of independent uniform variables. Then*

$$
\lim_{m\to\infty}\sup_{0\le x<1} \mid P\{\mu_k(m)/m \le x\} - P\{\Delta_{k,n} \le x\} \mid= 0,
$$

where $P\{\Delta_{k,n} \le x\}$ is given by (4).

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