



# Waiting times of exceedances in random threshold models

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## ABSTRACT

Waiting time distributions associated with a sequence of binary trials have been widely studied and applied in various areas of probability and statistics. In this paper we study simple and compound waiting time random variables associated with random threshold models which make the elements of sequence dependent. The exact distributions are derived and the results are illustrated for random thresholds based on order statistics.

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## 1. Introduction

Let  $\mathfrak{S}$  be a class of distribution functions (d.f.'s),  $X_1, X_2, \dots, X_n, X_{n+1}, \dots$  be a sequence of independent and identically distributed random variables with d.f.  $F \in \mathfrak{S}$ . Let  $f_1(u_1, u_2, \dots, u_n)$  and  $f_2(u_1, u_2, \dots, u_n)$  be two Borel functions satisfying

$$f_1(u_1, u_2, \dots, u_n) \leq f_2(u_1, u_2, \dots, u_n) \quad \forall (u_1, u_2, \dots, u_n) \in \mathbb{R}^n. \quad (1.1)$$

Denote  $\mathbf{J} \equiv \mathbf{J}(X_1, X_2, \dots, X_n) = (f_1(X_1, X_2, \dots, X_n), f_2(X_1, X_2, \dots, X_n))$ . We say that the random interval  $\mathbf{J}$  containing the future observation is invariant (or distribution free) with respect to the class  $\mathfrak{S}$  if the probability  $p = P\{X_{n+1} \in (f_1(X_1, X_2, \dots, X_n), f_2(X_1, X_2, \dots, X_n))\}$  is the same for all distributions from the class  $\mathfrak{S}$ . It is easy to show that the random interval  $(X_{r:n}, X_{s:n})$ ,  $1 \leq r < s \leq n$  form an invariant confidence interval containing the future observation for the class  $\mathfrak{S}_c$  of all continuous distribution functions with  $p = (s - r)/(n + 1)$ , where  $X_{r:n}$  and  $X_{s:n}$  are the  $r$ th and  $s$ th order statistics of the sample  $X_1, X_2, \dots, X_n$ . Let  $\mathbf{u} = (u_1, u_2, \dots, u_n)$  be any point of Euclidian space  $\mathfrak{R}^n$  and  $u_{(1)} \leq u_{(2)} \leq \dots \leq u_{(n)}$  be an arrangement of the coordinate of  $\mathbf{u}$  in the order of magnitude. Define the  $j$ th fundamental symmetric function as  $\psi_j(u_1, u_2, \dots, u_n) = u_{(j)}$ . Since any symmetric function satisfies  $\psi(u_1, u_2, \dots, u_n) = \psi(u_{i_1}, u_{i_2}, \dots, u_{i_n})$ , where  $(i_1, i_2, \dots, i_n)$  is any permutation of  $1, 2, \dots, n$ , it can be represented as a composite function of some function  $g(u_1, u_2, \dots, u_n)$  and fundamental symmetric functions  $\psi_1(u_1, u_2, \dots, u_n), \dots, \psi_n(u_1, u_2, \dots, u_n)$ , i.e.

$$\psi(u_1, u_2, \dots, u_n) = g(\psi_1(u_1, u_2, \dots, u_n), \dots, \psi_n(u_1, u_2, \dots, u_n)).$$

It is clear that the symmetric function  $\psi(u_1, u_2, \dots, u_n)$  coincides with one of the fundamental symmetric functions if and only if

$$\prod_{i=1}^n [\psi(u_1, u_2, \dots, u_n) - u_{(i)}] = 0.$$

It is known that if  $f_1$  and  $f_2$  are continuous and symmetric real functions of  $n$  variables satisfying (1.1) and if underlying distribution  $F$  is continuous then the interval  $(f_1(X_1, X_2, \dots, X_n), f_2(X_1, X_2, \dots, X_n))$  is an invariant interval containing the future observation if and only if  $f_1(u_1, u_2, \dots, u_n) = u_{(i)}$  and  $f_2(u_1, u_2, \dots, u_n) = u_{(j)}$  for some  $i < j$ , i.e. only the order statistics form unique invariant intervals for the class  $\mathfrak{S}_c$  (Bairamov and Petunin, 1991).

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Define the following binary random variables:

$$\xi_i = \begin{cases} 1 & X_{n+i} \in \mathbf{J} \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, 2, \dots$$

Let  $A_i \equiv \{\xi_i = 1\}$ , and  $A_i^c \equiv \{\xi_i = 0\}$  denote the events of “success” and “failure”, respectively. It is clear that  $\xi_1, \xi_2, \dots$  are dependent (exchangeable) random variables. Let  $S_m = \sum_{i=1}^m \xi_i$ , be the total number of observations  $X_{n+1}, X_{n+2}, \dots, X_{n+m}$  falling into the interval  $\mathbf{J}(X_1, X_2, \dots, X_n)$ . There are numerous papers devoted to the statistic  $S_m$  for different functions  $f_1$  and  $f_2$  and various classes of distributions. The particular cases for the lower and upper thresholds being the  $r$ th and  $s$ th order statistics,  $X_{r:n}$  and  $X_{s:n}$  from the sample  $X_1, X_2, \dots, X_n$  were investigated in connection with the theory of tolerance limits and invariant confidence intervals for future observations. See, for instance Robbins (1944), Gumbel and Schelling (1950), Epstein (1954). Some results on this topic are used to construct a statistical criteria for testing hypothesis  $H_0 : F = Q$  against some classes of alternatives, where  $Q$  is assumed to be an unknown c.d.f. of  $X_{n+1}, X_{n+2}, \dots, X_{n+m}$ . For example, Matveychuk and Petunin (1991) and Johnson and Kotz (1991) studied a generalized Bernoulli model defined in terms of placement statistics from two random samples. Katzenbeisser (1985) obtained a formula for the distribution of  $S_m$  when  $f_1(X_1, X_2, \dots, X_n) = -\infty$  and  $f_2(X_1, X_2, \dots, X_n) = X_{r:n}$  and proposed a test criterion for testing the null hypothesis  $H_0 : F(x) = Q(x)$  versus Lehmann alternatives  $Q(x) = [F(x)]^\theta$ ,  $\theta \neq 1$ . He extended these results to shift alternatives (Katzenbeisser (1986)). Matveychuk and Petunin (1991), Johnson and Kotz (1991, 1994) investigated the test criterion for testing the hypothesis  $H_0 : F(x) = Q(x)$  by using  $S_m$ . For the recent results one can see Bairamov (1997), Wesolowski and Ahsanullah (1998), Bairamov and Eryilmaz (2000), Bairamov and Kotz (2001), Stepanov (2004), Bairamov and Khan (2007), Bairamov and Eryilmaz (2006) and Bairamov and Eryilmaz (2008). For a review on the works devoted to the exceedance statistics we refer to Bairamov (2007).

Waiting time random variables associated with a sequence of binary trials are widely discussed in the literature. In the simplest case of independent identically distributed binary trials, it is well known that the total number of trials to achieve  $r$  successes follows a negative binomial distribution (geometric distribution for  $r = 1$ ). Many other waiting time random variables in a sequence consisting of two possible outcomes have been defined and their distributions derived in the literature. These waiting time random variables are closely related to runs, patterns and scans which are widely studied in the literature (see, e.g. Balakrishnan and Koutras (2002)). The resulting distributions are called generalized type geometric and negative binomial distributions which are of special importance in various applications including statistical process control and reliability. See, e.g. Fu and Lou (2003), Chakraborti et al. (2004), Chakraborti and Eryilmaz (2007) and Eryilmaz and Demir (2007).

In this paper, we first consider binary trials with outcomes  $A_i \equiv \{\xi_i = 1\}$ , and  $A_i^c \equiv \{\xi_i = 0\}$  and investigate the total number of trials  $Y_r$  to achieve  $r$ th success in  $\xi_1, \xi_2, \dots$ . It can be easily seen that the statistic  $S_m$  and waiting time  $Y_r$  obey the probability equality

$$P\{S_k \leq r\} = P\{Y_{r+1} \geq k + 1\}, \quad r \geq 1.$$

We also consider the compound waiting time random variable  $W_{r,s} \equiv \min(Y_r^{(0)}, Y_s^{(2)})$  defined in a sequence of trivalued random variables

$$\eta_i = \begin{cases} 0 & \text{if } X_{n+i} \leq f_1(X_1, X_2, \dots, X_n) \\ 1 & \text{if } X_{n+i} \in \mathbf{J} \equiv (f_1(X_1, X_2, \dots, X_n), f_2(X_1, X_2, \dots, X_n)) \\ 2 & \text{if } X_{n+i} \geq f_2(X_1, X_2, \dots, X_n), \end{cases}$$

$i = 1, 2, \dots$ , where  $Y_r^{(0)}$  and  $Y_s^{(2)}$  denote the waiting time for getting  $r$  0s and  $s$  2s in  $\eta_1, \eta_2, \dots$ , respectively. In our paper, the new discrete distributions are obtained for the particular cases when  $f_1$  and  $f_2$  are symmetric functions.

## 2. Distributions of waiting times

Let  $Y_r$  be the total number of trials to achieve  $r$ th success in  $\xi_1, \xi_2, \dots$ , i.e.  $Y_r = \min\{i : S_i \geq r\}$  is the waiting time for the  $r$ th success. If  $Z_1$  is the number of trials to achieve the first success,  $Z_2$  is the number of additional trials to achieve the second success etc., then  $Z_r$  can also be represented as  $Y_r = Z_1 + Z_2 + \dots + Z_r$ .

**Theorem 1.** *It is true that*

$$P\{Y_r = k\} = \binom{k-1}{r-1} \varphi_F([ \varphi_F(X_1, X_2, \dots, X_n) ]^r [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{k-r}),$$

$$k = r, r + 1, \dots, \quad (2.1)$$

where  $\varphi_F(u_1, u_2, \dots, u_n) = F(f_2(u_1, u_2, \dots, u_n)) - F(f_1(u_1, u_2, \dots, u_n))$ .

**Proof.** Since  $\xi_1, \xi_2, \dots$  are exchangeable, we have

$$\begin{aligned} P\{Y_r = k\} &= P\left\{\bigcup_{i_1, i_2, \dots, i_k} A_{i_1} A_{i_2} \dots A_{i_{r-1}} A_{i_r}^c \dots A_{i_{k-1}}^c A_k\right\} \\ &= \binom{k-1}{r-1} P\{X_{n+1} \in \mathbf{J}, \dots, X_{n+r-1} \in \mathbf{J}, X_{n+r} \notin \mathbf{J}, \dots, X_{n+k-1} \notin \mathbf{J}, X_{n+k} \in \mathbf{J}\} \\ &= \binom{k-1}{r-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} [F(f_2(u_1, u_2, \dots, u_n)) \\ &\quad - F(f_1(u_1, u_2, \dots, u_n))]^{r-1} [1 - (F(f_2(u_1, u_2, \dots, u_n)) \\ &\quad - F(f_1(u_1, u_2, \dots, u_n)))]^{k-r} [F(f_2(u_1, u_2, \dots, u_n)) - F(f_1(u_1, u_2, \dots, u_n))] \\ &\quad \times dF(u_1) dF(u_2) \dots dF(u_n) \\ &= \binom{k-1}{r-1} E_F[F(f_2(X_1, X_2, \dots, X_n)) - F(f_1(X_1, X_2, \dots, X_n))]^r \\ &\quad \times [1 - (F(f_2(X_1, X_2, \dots, X_n)) - F(f_1(X_1, X_2, \dots, X_n)))]^{k-r} \\ k &= r, r+1, \dots \end{aligned}$$

The theorem thus proved.  $\square$

**Remark 1.** In the case whenever the joint distribution of  $f_1(X_1, X_2, \dots, X_n)$  and  $f_2(X_1, X_2, \dots, X_n)$  is known the distribution of  $Y_r$  can also be formulated by the following simpler formula:

$$P\{Y_r = k\} = \binom{k-1}{r-1} \iint_{u_1 < u_2} (F(u_2) - F(u_1))^r (1 - (F(u_2) - F(u_1)))^{k-r} dH(u_1, u_2),$$

where  $H(u_1, u_2) = P\{f_1(X_1, X_2, \dots, X_n) \leq u_1, f_2(X_1, X_2, \dots, X_n) \leq u_2\}$ .  $\square$

It can be easily seen that the random variables  $Z_1, Z_2, \dots, Z_r$  are exchangeable and the joint distribution of these random variables is given by

$$P\{Z_1 = z_1, Z_2 = z_2, \dots, Z_r = z_r\} = E_F([\varphi_F(X_1, X_2, \dots, X_n)]^r [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{\sum_{j=1}^r z_j - r}).$$

**Proposition 1.** The expected value and the variance of  $Y_r$  are

$$\begin{aligned} E(Y_r) &= rE\left(\frac{1}{\varphi_F(\mathbf{X})}\right), \\ \text{Var}(Y_r) &= r\left[E\left(\frac{1}{\varphi_F^2(\mathbf{X})}\right) - E\left(\frac{1}{\varphi_F(\mathbf{X})}\right)^2\right] + r^2\text{Var}\left(\frac{1}{\varphi_F(\mathbf{X})}\right), \end{aligned}$$

where  $\mathbf{X} = (X_1, X_2, \dots, X_n)$ .

**Proof.** Through the representation  $Y_r = \sum_{j=1}^r Z_j$  we have

$$E(Y_r) = \sum_{j=1}^r E(Z_j),$$

and

$$\text{Var}(Y_r) = \sum_{j=1}^r E(Z_j^2) + 2 \sum_{1 \leq i < j \leq r} E(Z_i Z_j) - (E(Y_r))^2.$$

Since  $Z_j$ s are exchangeable

$$\begin{aligned} E(Z_j) &= \sum_{m=1}^{\infty} m E_F([\varphi_F(\mathbf{X})][1 - \varphi_F(\mathbf{X})]^{m-1}) = E\left(\frac{1}{\varphi_F(\mathbf{X})}\right), \\ E(Z_j^2) &= \sum_{m=1}^{\infty} m^2 E_F([\varphi_F(\mathbf{X})][1 - \varphi_F(\mathbf{X})]^{m-1}) = E\left(\frac{2 - \varphi_F(\mathbf{X})}{\varphi_F^2(\mathbf{X})}\right), \\ E(Z_i Z_j) &= \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} ml E_F([\varphi_F(\mathbf{X})]^2 [1 - \varphi_F(\mathbf{X})]^{m+l-2}) = E\left(\frac{1}{\varphi_F^2(\mathbf{X})}\right). \end{aligned}$$

Considering these moments in the formulas given above the proof is completed.  $\square$

**Remark 2.** If  $N_r$  is the waiting time for  $r$ th failure then a simple modification of Theorem 1 gives

$$P\{N_r = k\} = \binom{k-1}{r-1} E_F([\varphi_F(X_1, X_2, \dots, X_n)]^{k-r} [1 - \varphi_F(X_1, X_2, \dots, X_n)]^r). \quad (2.2)$$

**Remark 3.** Theorem 1 can be generalized as follows. Assume that  $X_1, X_2, \dots, X_n$  are i.i.d. r.v.'s with distribution function  $F$  and  $X'_1, X'_2, \dots, X'_m, \dots$  is a sequence of i.i.d. r.v.'s with d.f.  $Q$ . Let

$$\xi'_i = \begin{cases} 1 & X'_i \in J(X_1, X_2, \dots, X_n) \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, 2, \dots$$

and  $Y'_r$  be the total number of trials to achieve  $r$ th success in  $\xi'_1, \xi'_2, \dots$ . It is clear from Theorem 1 that the distribution of  $Y'_r$  is

$$P\{Y'_r = k\} = \binom{k-1}{r-1} E_F([\varphi_Q(X_1, X_2, \dots, X_n)]^r [1 - \varphi_Q(X_1, X_2, \dots, X_n)]^{k-r}),$$

$k = r, r + 1, \dots \quad \square$

Now consider the joint probability function of  $Y_1, Y_2, \dots, Y_r$ . First consider  $P\{Y_1 = k, Y_2 = l\}$ . It is clear

$$\begin{aligned} P\{Y_1 = k, Y_2 = l\} &= P\{X_1 \notin \mathbf{J}, \dots, X_{k-1} \notin \mathbf{J}, X_k \in \mathbf{J}, X_{k+1} \notin \mathbf{J}, \dots, X_{l-1} \notin \mathbf{J}, X_l \in \mathbf{J}\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} [1 - F(f_2(u_1, u_2, \dots, u_n)) + F(f_1(u_1, u_2, \dots, u_n))]^{k-1+l-k-1} \\ &\quad \times [F(f_2(u_1, u_2, \dots, u_n)) - F(f_1(u_1, u_2, \dots, u_n))]^2 dF(u_1) \dots dF(u_n) \\ &= E_F([\varphi_F(X_1, X_2, \dots, X_n)]^2 [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{l-2}), \\ &k = 1, 2, \dots, l - 1; \quad l = 2, 3, \dots \end{aligned}$$

Analogously,

$$\begin{aligned} P\{Y_1 = k, Y_2 = l, Y_3 = m\} &= E_F([\varphi_F(X_1, X_2, \dots, X_n)]^3 [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{m-3}), \\ &k = 1, 2, \dots, l - 1; \quad l = 2, 3, \dots, m - 1; \quad m = 3, 4, \dots \end{aligned}$$

By the induction we have the following:

**Theorem 2.** The joint probability function of  $Y_1, Y_2, \dots, Y_r$  is given by

$$\begin{aligned} P\{Y_1 = i_1, Y_2 = i_2, Y_3 = i_3, \dots, Y_r = i_r\} &= E_F([\varphi_F(X_1, X_2, \dots, X_n)]^r [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{i_r-r}), \\ i_1 = 1, 2, \dots, i_2 - 1; \quad i_2 = 2, 3, \dots, i_3 - 1; \quad \dots, \quad i_r = r, r + 1, \dots \quad \square \end{aligned}$$

**Theorem 3.** The sequence of waiting times  $Y_1, Y_2, \dots, Y_n, \dots$  is a Markov chain with transition probabilities

$$\begin{aligned} P\{Y_k = j \mid Y_{k-1} = i\} &= \frac{E_F([\varphi_F(X_1, X_2, \dots, X_n)]^k [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{j-k})}{E_F([\varphi_F(X_1, X_2, \dots, X_n)]^{k-1} [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{i-k+1})}. \end{aligned} \quad (2.3)$$

**Proof.** Consider

$$\begin{aligned} P\{Y_r = i_r \mid Y_1 = i_1, Y_2 = i_2, \dots, Y_{r-1} = i_{r-1}\} &= \frac{P\{Y_1 = i_1, Y_2 = i_2, Y_3 = i_3, \dots, Y_r = i_r\}}{P\{Y_1 = i_1, Y_2 = i_2, Y_3 = i_3, \dots, Y_{r-1} = i_{r-1}\}} \\ &= \frac{E_F([\varphi_F(X_1, X_2, \dots, X_n)]^r [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{i_r-r})}{E_F([\varphi_F(X_1, X_2, \dots, X_n)]^{r-1} [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{i_{r-1}-r+1})}. \end{aligned} \quad (2.4)$$

Now consider

$$\begin{aligned}
 P\{Y_r = i_r \mid Y_{r-1} = i_{r-1}\} &= \frac{P\{Y_{r-1} = i_{r-1}, Y_r = i_r\}}{P\{Y_{r-1} = i_{r-1}\}} \\
 &= \frac{\sum_{i_{r-2}=r-2}^{i_{r-1}-1} \cdots \sum_{i_2=2}^{i_3-1} \sum_{i_1=1}^{i_2-1} P\{Y_1 = i_1, Y_2 = i_2, Y_3 = i_3, \dots, Y_r = i_r\}}{P\{Y_{r-1} = i_{r-1}\}} \\
 &= \frac{\sum_{i_{r-2}=r-2}^{i_{r-1}-1} \cdots \sum_{i_2=2}^{i_3-1} \sum_{i_1=1}^{i_2-1} E_F[(\varphi_F(X_1, X_2, \dots, X_n))^r [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{i_r-r}]}{\binom{i_{r-1}-1}{r-2} E_F[(\varphi_F(X_1, X_2, \dots, X_n))^{r-1} [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{i_{r-1}-r+1}]} \tag{2.5}
 \end{aligned}$$

It can be easily shown that

$$\begin{aligned}
 \sum_{i_{r-2}=r-2}^{i_{r-1}-1} \cdots \sum_{i_2=2}^{i_3-1} \sum_{i_1=1}^{i_2-1} 1 &= \frac{(i_{r-1} - r + 2)(i_{r-1} - r + 3) \cdots (i_{r-1} - 1)}{(r - 2)!} \\
 &= \frac{(i_{r-1} - 1)!}{(r - 2)!(i_{r-1} - r + 1)!} = \binom{i_{r-1} - 1}{r - 2}. \tag{2.6}
 \end{aligned}$$

Proof of (2.6) can be easily made by induction. Indeed, for  $r = 3$ ,  $\sum_{i_1=1}^{i_2} 1 = (i_2 - 1)$  and  $r = 4$ ,  $\sum_{i_2=2}^{i_3} \sum_{i_1=1}^{i_2} 1 = \frac{(i_3-2)(i_3-1)}{2!}$ . Now assume that (2.6) is true for  $r = k$ . Then for  $r = k + 1$  we have

$$\sum_{i_{k-1}=k-1}^{i_k-1} \sum_{i_{k-2}=k-2}^{i_{k-1}-1} \cdots \sum_{i_2=2}^{i_3-1} \sum_{i_1=1}^{i_2-1} 1 = \sum_{i_{k-1}=k-1}^{i_k} \binom{i_{k-1} - 1}{k - 2} = \binom{i_k - 1}{k - 1}.$$

Now from (2.5), (2.6) and (2.4) we have (2.3).  $\square$

Since  $Y_1, Y_2, \dots, Y_n, \dots$  is a Markov chain the  $m$ -step transition probabilities  $P\{Y_t = j \mid Y_{t-m} = i\} = p_{ij}^{(m)}(t)$  can be obtained by an important identity known as the Chapman-Kolmogorov equation. If  $m = 2$ , we have

$$\begin{aligned}
 p_{ij}^{(2)}(t) &= \sum_{k=t-1}^{j-1} P\{Y_t = j \mid Y_{t-1} = k\} P\{Y_{t-1} = k \mid Y_{t-2} = i\} \\
 &= \sum_{k=t-1}^{j-1} p_{kj}(t) p_{ik}(t-1).
 \end{aligned}$$

The conditional expectation of  $Y_t$  given the value of  $Y_{t-m}$  can also be computed using these  $m$ -step transition probabilities. For  $m = 1$  we have

$$\begin{aligned}
 E(Y_t \mid Y_{t-1} = i) &= \sum_{j=t}^{\infty} j \cdot P\{Y_t = j \mid Y_{t-1} = i\} \\
 &= \frac{1}{E[(\varphi_F(\mathbf{X}))^{t-1} (1 - \varphi_F(\mathbf{X}))^{i-t+1}]} \sum_{j=t}^{\infty} j \cdot E[(\varphi_F(\mathbf{X}))^t (1 - \varphi_F(\mathbf{X}))^{j-t}] \\
 &= \frac{E[(\varphi_F(\mathbf{X}))^{t-2} (1 - \varphi_F(\mathbf{X})) + t \cdot \varphi_F(\mathbf{X})]}{E[(\varphi_F(\mathbf{X}))^{t-1} (1 - \varphi_F(\mathbf{X}))^{i-t+1}]}.
 \end{aligned}$$

For  $m = 2$

$$\begin{aligned}
 E(Y_t \mid Y_{t-2} = i) &= \sum_{j=t}^{\infty} \sum_{k=t-1}^{j-1} j \cdot p_{kj}(t) p_{ik}(t-1) \\
 &= \frac{1}{E[(\varphi_F(\mathbf{X}))^{t-2} (1 - \varphi_F(\mathbf{X}))^{i-t+2}]} \sum_{j=t}^{\infty} j(j-t+1) E[(\varphi_F(\mathbf{X}))^t (1 - \varphi_F(\mathbf{X}))^{j-t}] \\
 &= \frac{E[(\varphi_F(\mathbf{X}))^{t-3} (2 \cdot (1 - \varphi_F(\mathbf{X})) + t \cdot \varphi_F(\mathbf{X}))]}{E[(\varphi_F(\mathbf{X}))^{t-2} (1 - \varphi_F(\mathbf{X}))^{i-t+2}]}.
 \end{aligned}$$

By the induction we have the following:

**Theorem 4.** For  $i = t - m, t - m + 1, \dots$

$$E(Y_t | Y_{t-m} = i) = \frac{E[(\varphi_F(\mathbf{X}))^{t-m-1} (m \cdot (1 - \varphi_F(\mathbf{X})) + t \cdot \varphi_F(\mathbf{X}))]}{E[(\varphi_F(\mathbf{X}))^{t-m} (1 - \varphi_F(\mathbf{X}))^{i-t+m}]} \quad \square$$

It is well known that under the minimum of least square criterion the best possible predictor of the  $t$ th success  $Y_t$  given  $Y_{t-m}$  is  $E(Y_t | Y_{t-m})$ ,  $t = m + 1, m + 2, \dots; m = 1, 2, 3, \dots$  Theorem 4 allows describing the best predictor of  $Y_t$  if  $Y_{t-m}$  is already known.

### 3. Illustrative examples

In the following we compute the distribution of  $Y_r$  for various choices of  $f_1(X_1, X_2, \dots, X_n)$  and  $f_2(X_1, X_2, \dots, X_n)$ .

**1.**  $f_1(u_1, u_2, \dots, u_n) = -\infty$  and  $f_2(u_1, u_2, \dots, u_n) = u_{m:n}$ , where  $u_{1:n} \leq u_{2:n} \leq \dots \leq u_{n:n}$ . We have

$$\begin{aligned} E_F([\varphi_F(X_1, X_2, \dots, X_n)]^r [1 - \varphi_F(X_1, X_2, \dots, X_n)]^{k-r}) \\ = E_F(F^r(X_{m:n}) [1 - F(X_{m:n})]^{k-r}) \\ = \frac{1}{B(m, n - m + 1)} \int_0^1 x^r (1 - x)^{k-r} x^{m-1} (1 - x)^{n-m} dx \\ = \frac{B(r + m, k - r + n - m + 1)}{B(m, n - m + 1)} \end{aligned}$$

and from Theorem 1

$$P\{Y_r = k\} = \binom{k-1}{r-1} \frac{B(r + m, k - r + n - m + 1)}{B(m, n - m + 1)}.$$

Therefore

$$P\{Y_r = k\} = \frac{m \binom{k-1}{r-1} \binom{n}{m}}{(r + m) \binom{k+n}{r+m}}, \quad k = r, r + 1, \dots$$

The joint p.m.f of  $Y_1, Y_2, \dots, Y_m$  is

$$\begin{aligned} P\{Y_1 = i_1, \dots, Y_r = i_r\} &= \frac{B(r + m, i_r - r + n - m + 1)}{B(m, n - m + 1)} \\ i_1 &= 1, 2, \dots, i_2 - 1; i_2 = 1, 2, \dots, i_3 - 1; \dots, i_r = r, r + 1, \dots \end{aligned}$$

and the transition probabilities are

$$\begin{aligned} P\{Y_r = j | Y_{r-1} = i\} &= \frac{B(r + m, j - r + n - m + 1)}{B(r + m - 1, i - r + n - m + 2)} \\ i &= r - 1, r, \dots, j - 1; j = r, r + 1, \dots \end{aligned}$$

**2.**  $f_1(u_1, u_2, \dots, u_n) = u_{l:n}$  and  $f_2(u_1, u_2, \dots, u_n) = u_{m:n}$ ,  $1 \leq l < m \leq n$ , where  $u_{1:n} \leq u_{2:n} \leq \dots \leq u_{n:n}$ . Since  $F(X_{m:n}) - F(X_{l:n})$  has the same distribution as  $F(X_{m-l:n})$ , the distribution of  $Y_r$  is readily obtained substituting  $m$  by  $m - l$  in the above example. Thus we have

$$P\{Y_r = k\} = \frac{m - l}{r + m - l} \frac{\binom{k-1}{r-1} \binom{n}{m-l}}{\binom{k+n}{r+m-l}}, \quad k = r, r + 1, \dots$$

Similarly, the joint distribution and transition probabilities associated with  $\{Y_i\}_{i \geq 1}$  are obtained respectively as

$$\begin{aligned} P\{Y_1 = i_1, \dots, Y_r = i_r\} &= \frac{B(r + m - l, i_r - r + n - m + l + 1)}{B(m - l, n - m + l + 1)} \\ i_1 &= 1, 2, \dots, i_2 - 1; i_2 = 1, 2, \dots, i_3 - 1; \dots, i_r = r, r + 1, \dots \end{aligned}$$

and

$$\begin{aligned} P\{Y_r = j | Y_{r-1} = i\} &= \frac{B(r + m - l, j - r + n - m + l + 1)}{B(r + m - l - 1, i - r + n - m + l + 2)} \\ i &= r - 1, r, \dots, j - 1; j = r, r + 1, \dots \end{aligned}$$

#### 4. Compound waiting time

In the previous section the successively occurring events have been classified into two groups depending on whether or not the observation  $X_{n+i}$  falls into the interval  $\mathbf{J} \equiv (f_1(X_1, X_2, \dots, X_n), f_2(X_1, X_2, \dots, X_n))$ . In a random threshold model consisting of two thresholds it might be sometimes necessary to consider three disjoint regions or equivalently three disjoint events. This may help to understand the tendency (below the lower threshold or above the upper threshold) of the observations if they do not fall into the interval  $\mathbf{J}$ . In this framework let us define the following trivalued random variables:

$$\eta_i = \begin{cases} 0 & \text{if } X_{n+i} \leq f_1(X_1, X_2, \dots, X_n) \\ 1 & \text{if } X_{n+i} \in \mathbf{J} \equiv (f_1(X_1, X_2, \dots, X_n), f_2(X_1, X_2, \dots, X_n)) \\ 2 & \text{if } X_{n+i} \geq f_2(X_1, X_2, \dots, X_n), \end{cases}$$

$i = 1, 2, \dots$ . Let  $Y_r^{(0)}$  ( $Y_s^{(2)}$ ) denote the waiting time for getting  $r$  ( $s$ ) observations below (above) the threshold  $f_1(X_1, X_2, \dots, X_n)$  ( $f_2(X_1, X_2, \dots, X_n)$ ). Define

$$W_{r,s} \equiv \min \{Y_r^{(0)}, Y_s^{(2)}\}.$$

**Theorem 5.** *It is true that*

$$P \{W_{r,s} > k\} = \sum \sum \binom{k}{j_1, j_2, k-j_1-j_2} E [(F(f_1(X_1, X_2, \dots, X_n)))^{j_1} \times (F(f_2(X_1, X_2, \dots, X_n)))^{j_2} (\varphi_F(X_1, X_2, \dots, X_n))^{k-j_1-j_2}],$$

where the double sum is taken over the set  $\{(j_1, j_2) : j_1 + j_2 \leq k, j_1 < r, j_2 < s\}$ .

**Proof.** By the definition of  $W_{r,s}$

$$\begin{aligned} P \{W_{r,s} > k\} &= P \{Y_r^{(0)} > k, Y_s^{(2)} > k\} \\ &= P \{S_k^{(0)} < r, S_k^{(2)} < s\}, \end{aligned}$$

where  $S_k^{(0)}$  ( $S_k^{(2)}$ ) denotes the number of observations  $X_{n+1}, X_{n+2}, \dots, X_{n+m}$  below (above) the random threshold  $f_1(X_1, X_2, \dots, X_n)$  ( $f_2(X_1, X_2, \dots, X_n)$ ). Now using the multinomial distribution and conditioning on  $X_1, X_2, \dots, X_n$  one obtains

$$\begin{aligned} P \{W_{r,s} > k\} &= \sum_{j_1 < r} \sum_{j_2 < s} \binom{k}{j_1, j_2, k-j_1-j_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (F(f_1(u_1, u_2, \dots, u_n)))^{j_1} \\ &\quad \times (F(f_2(u_1, u_2, \dots, u_n)))^{j_2} (F(f_2(u_1, u_2, \dots, u_n)) - F(f_1(u_1, u_2, \dots, u_n)))^{k-j_1-j_2} dF(u_1)dF(u_2) \dots dF(u_n). \end{aligned}$$

The theorem thus proved.  $\square$

**Remark 4.** In the case whenever the joint distribution of  $f_1(X_1, X_2, \dots, X_n)$  and  $f_2(X_1, X_2, \dots, X_n)$  is known we have the following simpler formula:

$$P \{W_{r,s} > k\} = \sum_{j_1 < r} \sum_{j_2 < s} \binom{k}{j_1, j_2, k-j_1-j_2} \iint_{u_1 < u_2} (F(u_1))^{j_1} (1 - F(u_2))^{j_2} (F(u_2) - F(u_1))^{k-j_1-j_2} dH(u_1, u_2),$$

where  $H(u_1, u_2) = P \{f_1(X_1, X_2, \dots, X_n) \leq u_1, f_2(X_1, X_2, \dots, X_n) \leq u_2\}$ .  $\square$

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