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Some distribution free properties of statistics based on record values and characterizations of the distributions through a record

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Abstract

Let $X_1, X_2, ..., X_n, ...$ is a sequence of independent and identically distributed (i.i.d.) random variables (r.v.) with continuous distribution function (d.f.) F. Define a sequence of record times $U(n)$ as follows: $U(1) = 1$, $U(n) = \min\{j : j > U(n-1), \quad X_j > X_{U(n-1)}\}, n > 1$. Let $X_{U(n)}$ be upper record values, $n = 1, 2, ...$. Suppose that $X_1', X_2', ..., X_n', ...$ is an another sequence of i.i.d. r.v.-s with d.f. F. In this paper we investigate some statistics based on $X_{U(n)}$, $U(n)$ and $X_1', X_2', ..., X_n', ...$. A characterization of the uniform distribution is given based on $X_{U(n)}$.

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1. INTRODUCTION

Let $X_\infty = (X_1, X_2, ..., X_n, ...)$ be an infinite sample from continuous distributions with d.f. $F(x)$, $0 < F(x) < 1$. $X = [\,X_\infty]_n$ is the first n coordinates. As infinite sample, X_∞, we consider an element of sample space $(\Re^\infty, B^\infty, P^\infty)$, where $\Re^\infty$ is the space of $(X_1, X_2, ..., X_n, ...)$ sequences; the $\sigma-$algebra, B^∞, is a $\sigma-$algebra arising from sets $\bigcap\limits_{j \le N} \{X_j \in B_j\}$, $B_j \in B$, $N = 1, 2, ...$, such that B is the Borel $\sigma-$algebra over the subset of $\Re$; and P^∞ is a probability measure according to $F(x)$ distribution function on $(\Re^\infty, B^\infty)$. Here the symbol $[.]_n$ denotes the operator of projection from $\Re^\infty$ in $\Re^n$.

Let $U(n)$ and $X_{U(n)}$ be upper record times and record values , respectively. It is known that the d.f. of record value is (see Ahsanullah ,1995)

$$F_n(x) = P\left\{ X_{U(n)} \le x \right\} = \frac{1}{(n-1)!}\int_{-\infty}^{x}\left[\ln\frac{1}{1-F(u)}\right]^{n-1} dF(u), -\infty < x < \infty.$$

The details on the theory of records can be found in the works of Ahsanullah (1995), (1992), Galambos (1978), Nagaraja (1988), Nevzorov (1988), among others.

Suppose that $X'_\infty = (X'_1, X'_2, ..., X'_n, ...)$ be an another infinite sample from the distribution with d.f. F and X'_∞ is obtained independently from X_∞. Let $[X'_\infty]_m = (X'_1, X'_2, ..., X'_m)$. We are interested in the behavior of $[X'_\infty]_m$ and X'_∞ related to $X_{U(r)},\ r = 1, 2, ...$.

2. DISTRIBUTION FREE PROPERTIES

Consider $X_{U(1)},\ X_{U(2)}, ...,\ X_{U(n)}, ...$, and $X'_1, X'_2, ..., X'_m$.

Lemma 2.1. *For any $k = 1, 2, ..., m$ and $r = 1, 2, ...$ it is true that*

$$P\left\{X'_k < X_{U(r)}\right\} = 1 - \frac{1}{2^r}.$$

Proof..

$$P\left\{X'_k < X_{U(r)}\right\} = \frac{1}{(r-1)!}\int_{-\infty}^{\infty} F(u)\left[\ln\frac{1}{1-F(u)}\right]^{r-1} dF(u)$$

$$= \frac{1}{(r-1)!}\int_0^1 u\left[\ln\frac{1}{1-u}\right]^{r-1} du = \frac{1}{(r-1)!}\int_0^\infty y^{r-1}(1-e^{-y})e^{-y}dy = 1 - \frac{1}{2^r}. \tag{2.1}$$

Corollary 2.1. *For any $k, r = 1, 2, ...$ and $s > r$, it is true that*

$$P\left\{X_{U(r)} < X'_k < X_{U(s)}\right\} = \frac{1}{2^r} - \frac{1}{2^s}\,. \tag{2.2}$$

Let us define the following r.v. for given r :

$$\xi_i(r) = 1 \ \text{ if } X'_i < X_{U(r)} \text{ and } \xi_i(r) = 0 \ \text{ if } X'_i \ge X_{U(r)}\ ,\quad i = 1, 2, ..., m$$

and denote $S_m(r) = \sum_{i=1}^m \xi_i(r)$. It is clear that $S_m(r)$ is the number of observations $X'_1, X'_2, ..., X'_m$ which are less than $X_{U(r)}$. Note that the random variables $\xi_1(r), \xi_2(r), ..., \xi_m(r)$ are generally dependent, therefore, we do not have Bernoulli trials here.

Theorem 2.1. For any $m, r = 1, 2, ...$

$$P\left\{S_m(r) = k\right\} = \frac{\binom{m}{k}}{(r-1)!}\int_0^\infty e^{-z(m-k+1)}(1-e^{-z})^k z^{r-1} dz \tag{2.3}$$

$k = 0, 1, 2, ..., m.$

Proof. From the definition of $S_m(r)$ immediately follows

$$P\left\{S_m(r) = k\right\} = \sum_{i_1, i_2, ..., i_m} P\left\{A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k} \cap \bar{A}_{i_{k+1}} \cap \bar{A}_{i_{k+2}} \cap ... \cap \bar{A}_{i_m} \right\}, \tag{2.4}$$

where $A_{i_k} = \left\{X'_{i_k} < X_{U(r)}\right\}$ $k = 0, 1, 2, ..., m$ and $\bar{A}_{i_k}$ denotes the complements of event A_{i_k}. One has

$$P\left\{A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k} \cap \bar{A}_{i_{k+1}} \cap \bar{A}_{i_{k+2}} \cap ... \cap \bar{A}_{i_m} \right\} =$$

$$= P\left\{X'_{i_1} < X_{U(r)}, ..., X'_{i_k} < X_{U(r)}, X'_{i_{k+1}} \geq X_{U(r)}, ..., X'_{i_m} \geq X_{U(r)}\right\} =$$

$$= \frac{1}{(r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{v} \overset{(k)}{\cdots} \int_{-\infty}^{v} \int_{v}^{\infty} \overset{(m-k)}{\cdots} \int_{v}^{\infty} \left[\ln \frac{1}{1-F(v)}\right]^{r-1} dF(u_1)dF(u_2)...dF(u_m)dF(v)$$

$$= \frac{1}{(r-1)!} \int_{-\infty}^{\infty} (1-F(v))^{m-k} F^k(v) \left[\ln \frac{1}{1-F(v)}\right]^{r-1} dF(v)$$

$$= \frac{1}{(r-1)!} \int_{0}^{1} (1-y)^{m-k} y^k \left[\ln \frac{1}{1-y}\right]^{r-1} dy\,. \tag{2.5}$$

The number of summands in (2.4) is equal to $\binom{m}{k}$ and all of them have the same probability (2.5), hence

$$P\left\{S_m(r) = k\right\} = \frac{\binom{m}{k}}{(r-1)!} \int_{0}^{1} (1-y)^{m-k} y^k \left[\ln \frac{1}{1-y}\right]^{r-1} dy$$

$$= \frac{\binom{m}{k}}{(r-1)!} \int_{0}^{\infty} e^{-z(m-k+1)} (1-e^{-z})^k z^{r-1} dz\,. \quad (Q.E.D.)$$

3. THE ASYMPTOTIC DISTRIBUTIONS

Now, we investigate the behavior of the distribution, $S_m(r)$, for large m. First of all we compute the expected value and the variance of $S_m(r)$. By using Lemma 2.1 we have $E\xi_i(r) = 1 - \frac{1}{2^r}$, $i = 1, 2, ..., m$. Hence $ES_m(r) = m(1 - \frac{1}{2^r})$. For the variance of $S_m(r)$ we get

$$var((S_m(r)) = E(\sum_{i=1}^{m} \xi_i(r))^2 - m^2(1 - \frac{1}{2^r})^2 = \sum_{i=1}^{m} E(\xi_i(r))^2 + 2\sum_{i<j} E\xi_i(r)\xi_j(r)$$

$$-m^2(1 - \frac{1}{2^r})^2\,. \tag{3.1}$$

It is clear that

$$E(\xi_i(r))^2 = 1 - \frac{1}{2^r} \ . \ E(\xi_i(r))^2 = 1 - \frac{1}{2^r} \ . \tag{3.2}$$

Let us consider

$$E\xi_i(r)\xi_j(r) = P\left\{X_i' < X_{U(r)}, X_j' < X_{U(r)}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{v}\int_{-\infty}^{v} dF(u_1)dF(u_2)dF_r(v) =$$

$$= \frac{1}{(r-1)!}\int_{-\infty}^{\infty} F^2(v)\left[\ln\frac{1}{1-F(v)}\right]^{r-1} dF(v) = \frac{1}{(r-1)!}\int_0^1 x^2\left[\ln\frac{1}{1-x}\right]^{r-1} dx$$

$$= 1 - \frac{2}{2^r} + \frac{1}{3^r}.$$

Consequently we obtain

$$\sum_{i<j} E\xi_i(r)\xi_j(r) = \frac{m(m-1)}{2}\left[1 - \frac{2}{2^r} + \frac{1}{3^r}\right] \ . \tag{3.3}$$

By using (3.2) and (3.3) in (3.1) we obtain

$$var((S_m(r)) = m^2\left(\frac{1}{3^r} - \frac{1}{2^{2r}}\right) + m\left(\frac{1}{2^r} - \frac{1}{3^r}\right).$$

Let us denote

$$S_m^*(r) = \frac{S_m(r) - ES_m(r)}{\sqrt{var(S_m(r))}}. \text{ Then } \quad ES_m^*(r) = 0, \quad var(S_m^*(r)) = 1.$$

Denote $a = \frac{1}{2^r}$, $b = \sqrt{\frac{1}{3^r} - \frac{1}{2^{2r}}}$.

Theorem 3.1. *A statistic $S_m^*(r)$ has a continuous limiting distribution as $m \to \infty$, with probability density function f^* defined as follows*

$$f^*(x) = \begin{cases} \frac{b}{(r-1)!}\left[\ln\frac{1}{a-bx}\right]^{r-1} & if \quad x \in \left[\frac{a-1}{b}, \frac{a}{b}\right] \\ 0 & if \quad x \notin \left[\frac{a-1}{b}, \frac{a}{b}\right] \end{cases} .$$

Proof. Consider the characteristic function of $S_m(r)$:

$$\varphi_m(t) = E\exp(itS_m(r)) = \sum_{k=0}^{m} e^{itk} P\left\{S_m(r) = k\right\}$$

$$= \frac{1}{(r-1)!}\sum_{k=0}^{m} e^{itk}\binom{m}{k}\int_0^{\infty} e^{-z(m-k)}e^{-z}(1-e^{-z})^k z^{r-1}dz$$

$$= \frac{1}{(r-1)!}\int_0^{\infty} z^{r-1}e^{-z}\left\{\sum_{k=0}^{m}\binom{m}{k}\left[e^{it}(1-e^{-z})\right]^k e^{-z(m-k)}\right\} dz$$

$$= \frac{1}{(r-1)!}\int_0^{\infty} z^{r-1}e^{-z}\left[e^{it}(1-e^{-z}) + e^{-z}\right]^m dz$$

$$= \frac{1}{(r-1)!}\int_0^\infty z^{r-1}e^{-z}\left[1+(e^{it}-1)(1-e^{-z})\right]^m dz$$
$$= H(\left[1+(e^{it}-1)(1-e^{-z})\right]^m), \tag{3.4}$$

where
$$H(f) = \frac{1}{(r-1)!}\int_0^\infty z^{r-1}e^{-z}f(z)dz.$$

It is clear that the functional $H(f)$ has the following properties:

1. $H(1) = 1$
2. $H(c_1f_1 + c_2f_2) = c_1H(f_1) + c_2H(f_2)$, $c_1, c_2 = const.$

Denote $\varphi_m^*(t) = E\exp(itS_m^*(r))$. We clearly obtain

$$\varphi_m^*(t) = \exp(-\frac{itES_m(r)}{\sqrt{var(S_m(r))}})\varphi_m(\frac{t}{\sqrt{var(S_m(r))}}) = \exp(-\frac{itES_m(r)}{\sqrt{var(S_m(r))}})\times$$
$$\times H(\left[1+(\exp(\frac{it}{\sqrt{var(S_m(r))}})-1)(1-\exp(-z))\right]^m) = H(\exp(-\frac{itES_m(r)}{\sqrt{var(S_m(r))}})\times$$
$$\times\left[1+(\exp(\frac{it}{\sqrt{var(S_m(r))}})-1)(1-\exp(-z))\right]^m). \tag{3.5}$$

Denote

$$g_m(t) = \exp(-\frac{itES_m(r)}{\sqrt{var(S_m(r))}})\left[1+(\exp(\frac{it}{\sqrt{var(S_m(r))}})-1)(1-\exp(-z))\right]^m$$

$$\ln g_m(t) = -\frac{itES_m(r)}{\sqrt{var(S_m(r))}})+m\ln\left[1+(\exp(\frac{it}{\sqrt{var(S_m(r))}})-1)(1-\exp(-z))\right]$$
$$= -\frac{itES_m(r)}{\sqrt{var(S_m(r))}}+m\ln\left[1+(\frac{it}{\sqrt{var(S_m(r))}}+o(\frac{t}{\sqrt{var(S_m(r))}}))(1-\exp(-z))\right]$$
$$= -\frac{itES_m(r)}{\sqrt{var(S_m(r))}}+\frac{itm(1-\exp(-z))}{\sqrt{var(S_m(r))}}+O(\frac{1}{m}).$$

As a result it becomes from above as

$$g_m(t) = \exp(it\frac{m(1-\exp(-z))-ES_m(r)}{\sqrt{var(S_m(r))}})+O(\frac{1}{m}). \tag{3.6}$$

By taking (3.6) into consideration in (3.5) we have

$$H(g_m(t)) = H(\exp(it\frac{m(1-\exp(-z))-ES_m(r)}{\sqrt{var(S_m(r))}}))+O(\frac{1}{m}) \quad ,$$

and by having the limit

$$\lim_{m\to\infty}\varphi_m^*(t) = \lim_{m\to\infty}H(g_m(t)) = \lim_{m\to\infty}H(\exp(it\frac{m(1-\exp(-z))-m(1-2^{-r})}{\sqrt{m^2(\frac{1}{3^r}-\frac{1}{2^{2r}})+m(\frac{1}{2^r}-\frac{1}{3^r})}})) =$$

$$= H(\exp(it\frac{a-e^{-z}}{b})). \tag{3.7}$$

Let us rename $\varphi(t) = \lim_{m\to\infty} \varphi_m^*(t)$. One can prove that $\varphi(t)$ is continuous at the point $t = 0$. In fact, by using the expansion

$$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$$

one obtains from (3.7)

$$\varphi(t) = \exp(it\frac{a-e^{-z}}{b}) = 1 + \frac{a-e^{-z}}{b}it - \frac{(a-e^{-z})^2}{2b^2}t^2 + o(t^2)\ . \tag{3.8}$$

From (3.7) and (3.8) it follows

$$\varphi(t) = 1 + \frac{itH(a-e^{-z})}{b} - \frac{H((a-e^{-z})^2)}{2b^2}t^2 + o(t^2).$$

It is easy to see that $H(2^{-r} - e^{-z}) = 0,\ H((2^{-r} - e^{-z})^2) = b^2$. Hence

$$\varphi(t) = 1 - \frac{t^2}{2} + o(t^2).$$

Thus, if $t \to 0$, then

$$|\varphi(t) - \varphi(0)| = \left|1 - \frac{t^2}{2} + o(t^2) - 1\right| = \frac{t^2}{2} + o(t^2) \to 0.$$

Let $F_m(x)$ be the d.f. of the statistic $S_m^*(r)$, where

$$x = \frac{k - E(S_m^*(r))}{\sqrt{var(S_m^*(r))}}\ , \qquad k = 0, 1, 2, ..., m.$$

Using the Levy-Cramer theorem for characteristic functions (see Petrov ,1975, Theorem 10, P.15) we obtain that $F_m(x) \to F(x)$ as $m \to \infty, x \in \left[\frac{a-1}{b}, \frac{a}{b}\right]$ and F has a characteristic function $\varphi(t)$, that is

$$\varphi(t) = \int_{\frac{a-1}{b}}^{\frac{a}{b}} e^{itx} dF(x).$$

In other hand if we change the variable $\frac{a-\exp(-z)}{b} = x$ in (3.7) we have

$$\varphi(t) = H(e^{it\frac{a-\exp(-z)}{b}}) = \frac{1}{(r-1)!}\int_0^\infty z^{r-1}e^{-z}e^{it\frac{a-\exp(-z)}{b}}dz$$

$$= \frac{b}{(r-1)!}\int_{\frac{a-1}{b}}^{\frac{a}{b}} \left[\ln\frac{1}{a-bx}\right]^{r-1} e^{itx}dx.$$

Hence $F(x)$ has the density function

$$f^*(x) = \frac{b}{(r-1)!}\left[\ln\frac{1}{a-bx}\right]^{r-1} \quad \text{if} \quad x \in \left[\frac{a-1}{b}, \frac{a}{b}\right] \quad \text{and} \quad f^*(x) = 0 \text{ if } \ x \notin \left[\frac{a-1}{b}, \frac{a}{b}\right].$$

The following theorem clearly may be obtained from Theorem 3.1. However we adduce a different proof which is interesting in our opineon.

Theorem 3.2. *It is true that ,*

$$\lim_{m\to\infty}\sup_{0\le x\le 1}\left|P\left\{\frac{S_m(r)}{m}\le x\right\} - \frac{1}{(r-1)!}\int_0^x \left[\ln(\frac{1}{1-u})\right]^{r-1} du\right| = 0.$$

Proof. We have

$$S_m(r) = \sum_{i=1}^m \xi_i(r) = \sum_{i=1}^m I_{\{(-\infty, X_{U(r)})\}}(X_i'), \tag{3.9}$$

where $I_{\{A\}}(x) = 1 \ if \ x \in A \ \ and \ \ I_{\{A\}}(x) = 0 \ \ if \ \ x \notin A$. Using the representation (3.9) we may write

$$P\left\{\frac{S_m(r)}{m}\le x\right\} = P\left\{\frac{1}{m}\sum_{i=1}^m I_{\{(-\infty, X_{U(r)})\}}(X_i') \le x\right\}$$

$$= P\left\{\int_{-\infty}^{\infty} I_{\{(-\infty, X_{U(r)})\}}(u)dF_m^*(u) \ \le x\right\}, \tag{3.10}$$

where $F_m^*(u)$ denotes the empirical distribution function of sample $X_1', X_2', ..., X_m'$. Note that any infinite sample $X_\infty = (X_1, X_2, ..., X_n, ...)$ may be considered also as a sequence of i.i.d. random variables $X_1(\omega), X_2(\omega), ..., X_n(\omega), ...$ defined in probability space $\{\Omega, \Im, P\}$, where Ω is a set of points, $\Im$ is a $\sigma-$field of subsets of Ω, and P is a probability distribution of the elements of $\Im$, $F(x) = P\{\omega : X_i(\omega) \le x\}$. Denote

$$G^*(F) = \int_{-\infty}^{\infty} I_{\{(-\infty, x)\}}(u)dF(u)\,, \quad (x \text{ is fixed}) \tag{3.11}$$

$$G(F) = \int_{-\infty}^{\infty} I_{\{(-\infty, X_{U(r)})\}}(u)dF(u)\,, \tag{3.12}$$

where $G(F) = G(F)(\omega)$ is a random variable defined in the probability space $\{\Omega, \Im, P\}$. Using (3.12), one can write (3.10) as follows

$$P\left\{\frac{S_m(r)}{m}\le x\right\} = P\left\{\int_{-\infty}^{\infty} I_{\{(-\infty, X_{U(r)})\}}(u)dF_m^*(u) \ \le x\right\} = P\left\{G(F_m^*) \le x\right\}.$$

The functional (3.11) is continuous according to uniform metric. One can follow from the Glivenko-Cantelli Theorem $(P\left\{\omega : \sup_u |F_m^*(u) - F(u)| \to 0\right\} = 1,)$ $G^*(F_m^*) \to G^*(F)$ almost sure (see Borovkov,1984). It is clear that

$$P\left\{\omega : \lim_{m\to\infty} G(F_m^*) = G(F)\right\}$$

$$= \int_{-\infty}^{\infty} P\left\{\lim \int_{-\infty}^{\infty} I_{\{(-\infty, X_{U(r)})\}}(u) dF_m^*(u) = \int_{-\infty}^{\infty} I_{\{(-\infty, X_{U(r)})\}}(u) dF(u) / X_{U(r)} = x\right\} dF_r(x)$$

$$= \int_{-\infty}^{\infty} P\left\{\lim \int_{-\infty}^{\infty} I_{\{(-\infty, x)\}}(u) dF_m^*(u) = \int_{-\infty}^{\infty} I_{\{(-\infty, x)\}}(u) dF(u)\right\} dF_r(x)$$

$$= \int_{-\infty}^{\infty} P\left\{\lim G^*(F_m^*) = G^*(F)\right\} dF_r(x) = 1,$$

where $F_r(x) = P\left\{X_{U(r)} \leq x\right\}$. So $G(F_m^*) \to G(F)$ almost sure in $(\Omega, \Im, P)$. Thus, $G(F_m^*) \to G(F)$ in distribution. We have

$$P\left\{G(F) \leq x\right\} = P\left\{\int_{-\infty}^{\infty} I_{\{(-\infty, X_{U(r)})\}}(u) dF(u) \leq x\right\} = P\left\{\int_{-\infty}^{X_{U(r)}} dF(u) \leq x\right\}$$

$$= P\left\{F(X_{U(r)}) \leq x\right\} = \frac{1}{(r-1)!}\int_0^x \left[\ln(\frac{1}{1-u})\right]^{r-1} du.$$

Above uniform convergence is the consequence of convergence in distribution and continuity of d.f. $U_r(x) \equiv \frac{1}{(r-1)!}\int_0^x \left[\ln(\frac{1}{1-u})\right]^{r-1} du$.

Remark 3.1. Let $r = 1$, $X_{U(1)} = X_1$.Then $\xi_i(1) = 1$ if $X_i' < X_1$ and $\xi_i(1) = 0$ if $X_i' \geq X_1$, $i = 1, 2, ..., m$. And $S_m(1) = \sum_{i=1}^m \xi_i(1)$ is the number of observations $X_1', X_2', ..., X_m'$ which are less than X_1. From Theorem 3.2. follows that

$$\lim_{m\to\infty} \sup_{0\leq x\leq 1}\left|P\left\{\frac{S_m(1)}{m} \leq x\right\} - x\right| = 0 .$$

Remark 3.2. We may similarly obtain the following result for order statistics. Let $X_1, X_2, ..., X_n$ be a sample from the distribution with continuous d.f. F and $Y_1, Y_2, ..., Y_m$ be a sample from the distribution with continuous d.f. G. Also let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be order statistics constructed by $X_1, X_2, ..., X_n$. Consider the hypothesis $H_0 : F(u) = G(u)$. Let $\xi_i = 1$ if $X_{(r)} \leq Y_i \leq X_{(s)}$ and $\xi_i = 0$ if $X_{(r)} > Y_i \vee X_{(s)} < Y_i$, $i = 1, 2, ..., m$, $1 \leq r < s \leq n$. Denote $\nu_m = \sum_{i=1}^m \xi_i$.

Theorem 3.3. *If H_0 is true, then*

$$\sup_x \left|P\left\{\frac{\nu_m}{m} \leq x\right\} - P\left\{W_{rs} \leq x\right\}\right| \to 0, \text{ as } m \to \infty,$$

where $W_{rs} = F(X_{(s)}) - F(X_{(r)})$.

It is known that W_{rs} has the probability density function (see David, 1970)

$$f(w_{rs}) = \begin{cases} \frac{1}{B(s-r,n-s+r+1)} w_{rs}^{s-r-1}(1-w_{rs})^{n-s+r} & if \quad 0 \leq w_{rs} \leq 1 \\ 0 & otherwise \end{cases} .$$

4. CHARACTERIZATION OF UNIFORM DISTRIBUTION BY RECORDS

Let X_∞ be an infinite sample from the distribution with d.f. $F(x)$, X' denotes one observation from $F(x)$ and X' does not depend on X_∞. Denote by Λ a class of continuous distribution functions $F(x), -\infty < x < \infty$ with property

$$F(x) \geq x\,,\ 0<x<1 \quad or \quad F(x) \leq x\,, 0<x<1\,.$$

For example let

$$F_n(x) = \begin{cases} 0 & if \quad x \leq 0 \\ x^n & if \quad 0<x<1 \\ 1 & if \quad x \geq 1 \end{cases}, \quad G_n(x) = \begin{cases} 0 & if \quad x \leq 0 \\ 1-(1-x)^n & if \quad 0<x<1 \\ 1 & if \quad x \geq 1 \end{cases},$$

$n = 1, 2, \ldots$. Denote by $U_{a,b}(x)$ the d.f. of uniform distribution on $[a, b]$. It is clear that $F_n(x) \in \Lambda$, $G_n(x) \in \Lambda$, $n = 1, 2, \ldots$ and $U_{a,a+1}(x) \in \Lambda$, $a \in R$.

Theorem 4.1. *Let $F(x) \in \Lambda$. Then $F(x)$ is same with uniform distribution on $[0,1]$ if and only if*

$$EX_{U(n)} = 1 - 2^{-n} \quad \text{for some } n \geq 1. \tag{4.1}$$

Proof. Let $F(x) = U_{0,1}(x)$. Clearly one can write

$$EX_{U(n)} = \frac{1}{(n-1)!}\int_0^1 u\left[\ln(\frac{1}{1-u})\right]^{n-1} du = 1 - \frac{1}{2^n}.$$

To proof the rest of the Theorem suppose that $EX_{U(n)} = 1 - 2^{-n}$ for some $n \geq 1$. From the Lemma 2.1 it follows

$$1 - \frac{1}{2^n} = P\left\{X' < X_{U(n)}\right\} = \frac{1}{(n-1)!}\int_{-\infty}^{\infty} F(x)\left[\ln(\frac{1}{1-F(x)})\right]^{n-1} dF(x)$$

It follows from (4.1) for some $n \geq 1$

$$\frac{1}{(n-1)!}\int_{-\infty}^{\infty} x\left[\ln(\frac{1}{1-F(x)})\right]^{n-1} dF(x) - \frac{1}{(n-1)!}\int_{-\infty}^{\infty} F(x)\left[\ln(\frac{1}{1-F(x)})\right]^{n-1} dF(x) =$$

$$= \frac{1}{(n-1)!}\int_{-\infty}^{\infty}(x - F(x))\left[\ln\frac{1}{1-F(x)}\right]^{n-1} dF(x) = 0. \tag{4.2}$$

It follows with the assumption on F that $F(x) - x$ has the same sign in the above integral within the integration limits. Let $F(x) \geq x, 0 < x < 1$. Then $F(x) = 1$ if $x \geq 1$. Therefore one can write the following expression with the help of (4.2)

$$\frac{1}{(n-1)!}\int_{-\infty}^{1}(x - F(x))\left[\ln\frac{1}{1-F(x)}\right]^{n-1} dF(x)$$

$$+\frac{1}{(n-1)!}\int_0^1 (x-F(x))\left[\ln\frac{1}{1-F(x)}\right]^{n-1} dF(x)=0.$$

Hence $F(x)=x \;\; if \;\; 0<x<1$ and $F(x)=0 \;\; if \;\; x\le 0$. The case $F(x)\le x$, $0<x<1$ is investigated analoguously. (Q.E.D.) .

5. APPENDIX

In this section we are giving some results in order to explain why we are considering two independent samples X_∞ and X'_∞ from the same distribution with the d.f. F. Let us define $X_\infty=(X_1,X_2,...,X_n,...)$ and $X_{U(r)}$ be the r th upper record value. $X_{U(r)+1},X_{U(r)+2},...,\ X_{U(r)+m}$ are observations which comes after $X_{U(r)}$. It is not difficult to see that, random variables $X_{U(r)+1}$, $X_{U(r)+2},...,\ X_{U(r)+m},...$ are mutually independent and identically distributed with d.f. F, for any r . Furthermore $X_{U(r)+k}$ and $X_{U(r)}$ are independent for $k=1,2,...$ In fact, (for simplisity, consider the case of $r=2$)

$$P\left\{X_{U(r)+1}\ \le t\right\}=\sum_{i=2}^{\infty}P\left\{X_{U(r)+1}\ \le t,U(2)=i\right\}$$

$$=\sum_{i=2}^{\infty}P\left\{X_{i+1}\ \le t,X_2\le X_1,...,X_{i-1}\le X_1,X_i>X_1\right\}$$

$$=\sum_{i=2}^{\infty}P\left\{X_{i+1}\ \le t\right\}P\left\{X_2\le X_1,...,X_{i-1}\le X_1,X_i>X_1\right\}$$

$$=P\left\{X_{i+1}\ \le t\right\}\sum_{i=2}^{\infty}P\left\{U(2)=i\right\}=P\left\{X_{i+1}\ \le t\right\}=F(t).$$

Then from the Lemma 2.1. we have

$$P\left\{X_{U(r)+1}\ <X_{U(r)}\right\}=1-\frac{1}{2^r}\,.$$

With the analogy let $X_{U(r)+1},X_{U(r)+2},...,X_{U(r)+m}$ are observations which comes after $X_{U(r)}$. Then

$$P\left\{X_{U(r)+1}\ <X_{U(r)},...,X_{U(r)+m}\ <X_{U(r)}\right\}=P\left\{S_m(r)=m\right\}=\sum_{i=0}^{m}\binom{m}{i}(-1)^i\frac{1}{(i+1)^r}\,.$$

($S_m(r)$ is defined in section 1).

Remark 5.1. It is clear that in all theorems above we may replace X'_i by $X_{U(r)+i}$.

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