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ON THE CHARACTERISTIC PROPERTIES OF EXPONENTIAL DISTRIBUTION

ISMIHAN G. BAIRAMOV

Department of Statistics, Ankara University, 06100, Tandogan, Ankara, Turkey

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Abstract. New characterizations for the exponential distribution are given in terms of record values and the probabilities of finite sums of independent and identically distributed nonnegative random variables provided that the underlying distribution is either new better than used or new worse than used.

Key words and phrases: Characterization, exponential distribution, order statistics, record value, new better than used.

1. Introduction

The characterization properties for the exponential distribution has been studied by many authors. Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be order statistics of n independent, identically distributed (i.i.d.) nonnegative random variables (r.v.'s) $X_1, X_2, \ldots, X_n$ with common continuous distribution function (d.f) F. A well known important property is stated as follows: the random variables X_1 and $nX_{1:n}$ are identically distributed for all $n \geq 1$ iff F is a d.f. of the exponential law. For more details of known extensions of this result one can see monographs by Galambos and Kotz (1978), Azlarov and Volodin (1986) among many others. Various characterizations of the exponential distribution are given by the distributional properties of linear functions of order statistics and record values. Since there is a lot of research on this topic, it is difficult to list all of the results appeared in the literature. However, a good list of review of the characterization theorems for exponential distribution can be found in Arnold and Huang (1995).

The important class of life distributions is the class of new better than used (NBU) distributions defined as $\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y)$ where $\bar{F}(x) = 1 - F(x)$, x and y are nonnegative real numbers. F is said to be new worse than used (NWU) if the inequality is reversed. Note that the exponential distribution is the only distribution having the equality $\bar{F}(x+y) = \bar{F}(x)\bar{F}(y)$, $x \geq 0$, $y \geq 0$. We will say that F belongs to the class $\Im_1$, if F is either NBU or NWU.

In many papers exponential distribution has been characterized in the class of distributions $\Im_1$. For further details, one can see Krishnaji (1971), Ahsanullah (1977), Ramachandran (1979), Shimizu (1979), Huang (1981), Xu and Yang (1995).

The object of this paper is to prove some characterizations of the exponential distribution through properties of $X_1 + X_2 + \cdots + X_n$ and X_{n+1}, where $X_1, X_2, \ldots, X_n$, X_{n+1} are nonnegative i.i.d. r.v.'s with continuous d.f. F. We also give some characterizations for the exponential distribution through the properties of $X_{U(n)}$ and $X_{U(n)+1}$, n-th record value of the sequence of i.i.d. r.v.'s and the next observation which comes after n-th record, accordingly.

2. Characterizations

Let X be a nonnegative continuous random variable with distribution function F, and a be some finite positive number. We will say that F belongs to the class $\Im_a$, if for all $x \geq 0$, $y \geq 0$ either

$$\bar{F}(a(x+y)) \geq (\bar{F}(x))^a(\bar{F}(y))^a \quad \text{or} \quad \bar{F}(a(x+y)) \leq (\bar{F}(x))^a(\bar{F}(y))^a, \tag{2.1}$$

where $\bar{F}(x) = 1 - F(x)$.

If $a = 1$ then F is either NBU or NWU.

LEMMA 2.1. *Let $G(x)$, $x \geq 0$ be monotonic and nonnegative, $G(0) = 1$, $a > 0$. Suppose for all $x_1 \geq 0$, $x_2 \geq 0$*

$$G(a(x_1 + x_2)) = (G(x_1))^a(G(x_2))^a. \tag{2.2}$$

Then, if $G(x)$ is not identically zero or one, for all $x \geq 0$, $G(x) = e^{\lambda x}$ with some real number λ. Here $\lambda > 0$ if $G(x)$ is increasing and $\lambda < 0$ if $G(x)$ is decreasing.

PROOF. This lemma is an obvious reformulation of Cauchy's equation. In fact taking in (2.2) $x_2 = 0$ and using $G(0) = 1$ one has $G(ax_1) = G^a(x_1)$ for any $x_1 \geq 0$. Let $H(x) = G^a(x)$. Then (2.2) becomes

$$H(x_1 + x_2) = G^a(x_1 + x_2) = G(a(x_1 + x_2)) = G^a(x_1)G^a(x_2) = H(x_1)H(x_2),$$

so $H(x)$ as well as $G(x)$ is exponential.

Let $X_1, X_2, \ldots, X_n, X_{n+1}, \ldots$ be a sequence of independent and identically distributed random variables with distribution function F, and let a be some positive real number. Consider the sequence of r.v.'s $\xi_1(a), \xi_2(a), \ldots$ defined as follows:

$$\xi_n(a) = \begin{cases} 1 & \text{if} \quad X_{n+1} < a\sum_{i=1}^{n} X_i \\ 0 & \text{otherwise} \end{cases}, \qquad n = 1, 2, \ldots.$$

THEOREM 2.1. *Let X be a nonnegative r.v. having a continuous d.f. F satisfying $\inf\{x : F(x) > 0\} = 0$. Then the following statements are equivalent.*

(a) *X has an exponential distribution with density*

$$f_\theta(x) = \begin{cases} \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right) & \textit{if} \quad x \geq 0 \\ 0 & \textit{otherwise} \end{cases}, \qquad \theta > 0 \tag{2.3}$$

(b) *for some $n > 1$ $E\xi_n(a) = 1 - \frac{1}{(a+1)^n}$ and F belongs to class $\Im_a$.*

PROOF. Let $F(x) = 1 - \exp(-\frac{x}{\theta})$, $x \geq 0$, $\theta > 0$. Consider

$$\begin{aligned} E\xi_n(a) &= P\{X_{n+1} < a(X_1 + X_2 + \cdots + X_n)\} \\ &= P\left\{\frac{X_{n+1}}{\theta} < a\left(\frac{X_1}{\theta} + \frac{X_2}{\theta} + \cdots + \frac{X_n}{\theta}\right)\right\} \\ &= P\{Y_{n+1} < a(Y_1 + Y_2 + \cdots + Y_n)\}, \end{aligned} \tag{2.4}$$

where $Y_1, Y_2, \ldots, Y_n, Y_{n+1}$ are i.i.d. r.v.'s with d.f. $F_0(x)$. It is well known that $Y_1 + Y_2 + \cdots + Y_n$ has a gamma distribution with probability density function (p.d.f.)

$$g_{1,n}(x) = \begin{cases} \dfrac{1}{(n-1)!} x^{n-1} e^{-x}, & \text{if} \quad x \geq 0 \\ 0 & \text{eitherwise.} \end{cases}$$

Then we can write

$$\begin{aligned} (2.5) \qquad P\{Y_{n+1} < a(Y_1 + Y_2 + \cdots + Y_n)\} &= \int_0^\infty P\{Y_{n+1} < at\}\, g_{1,n}(t)dt \\ &= \frac{1}{(n-1)!} \int_0^\infty (1 - e^{-at}) e^{-t} t^{n-1} dt \\ &= 1 - \frac{1}{(a+1)^n}. \end{aligned}$$

Writing (2.5) in (2.4) we obtain $E\xi_n(a) = 1 - \frac{1}{(a+1)^n}$. Therefore $(a) \Rightarrow (b)$.

Now let $F \in \Im_a$ and $E\xi_n(a) = 1 - \frac{1}{(a+1)^n}$ for some $n > 1$. One can write

$$\begin{aligned} (2.6) \quad & P\{X_{n+1} < a(X_1 + \cdots + X_n)\} \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty F(a(x_1 + \cdots + x_n)) dF(x_1) \cdots dF(x_n) \\ &= \int_0^1 \int_0^1 \cdots \int_0^1 F(a(F^{-1}(x_1) + F^{-1}(x_2) + \cdots + F^{-1}(x_n))) dx_1 dx_2 \cdots dx_n \\ &= 1 - \frac{1}{(a+1)^n} \end{aligned}$$

where $F^{-1}(t) = \inf\{x : F(x) \geq t\}$ is the inverse function of F. On the other hand for $Y_1, Y_2, \ldots Y_n, Y_{n+1}$, using (2.5) and (2.6) we have

$$\begin{aligned} (2.7) \quad & P\{Y_{n+1} < a(Y_1 + Y_2 + \cdots + Y_n)\} \\ &= \int_0^1 \int_0^1 \cdots \int_0^1 (1 - e^{-a \sum_{i=1}^n - \ln(1-x_i)}) dx_1 dx_2 \cdots dx_n \\ &= \int_0^1 \int_0^1 \cdots \int_0^1 (1 - (1-x_1)^a (1-x_2)^a \cdots (1-x_n)^a) dx_1 dx_2 \cdots dx_n. \end{aligned}$$

From (2.6) and (2.7) one obtain

$$\begin{aligned} (2.8) \quad & \int_0^1 \cdots \int_0^1 [F(a(F^{-1}(x_1) + \cdots + F^{-1}(x_n))) - (1 - (1-x_1)^a \cdots (1-x_n)^a)] \\ & \times dx_1 \cdots dx_n = 0. \end{aligned}$$

Since $F \in \Im_a$, from (2.1) by mathematical induction one obtain for all $x_1 \geq 0$, $x_2 \geq 0, \ldots, x_n \geq 0$, $n > 1$

$$(2.9) \quad \begin{aligned} & F(a(x_1 + x_2 + \cdots + x_n)) \geq 1 - (1 - F(x_1))^a (1 - F(x_2))^a \cdots (1 - F(x_n))^a \quad \text{or} \\ & F(a(x_1 + x_2 + \cdots + x_n)) \leq 1 - (1 - F(x_1))^a (1 - F(x_2))^a \cdots (1 - F(x_n))^a. \end{aligned}$$

(2.9) implies

$$F(a(F^{-1}(x_1)+\cdots+F^{-1}(x_n))) \geq (\text{or} \leq) 1-(1-x_1)^a\cdots(1-x_n)^a, \quad x_i \in [0,1], \quad i=1,2,\ldots,n. \tag{2.10}$$

Taking (2.10) into consideration in (2.8) one obtains

$$F(a(F^{-1}(x_1)+\cdots+F^{-1}(x_n))) = 1-(1-x_1)^a\cdots(1-x_n)^a, \quad x_i \in [0,1], \quad i=1,2,\ldots,n$$

which is equivalent to

$$F(a(x_1+x_2+\cdots+x_n)) = 1-(1-F(x_1))^a(1-F(x_2))^a\cdots(1-F(x_n))^a \quad x_i \in [0,\infty), \quad i=1,2,\ldots,n$$

and from Lemma 2.1, $F(x)$ is an exponential distribution function.

Remark 2.1. (Counter example) The assumption $F \in \Im_1$ is needed in (b). Let $a=1$. Then Theorem 2.1 says that F is the exponential distribution function if and only if for some $n>1$

$$P\left\{X_{n+1} < \sum_{i=1}^{n} X_i\right\} = 1-\frac{1}{2^n} \tag{2.11}$$

and F belongs to the class $\Im_1$. Here is an example when $F \notin \Im_1$ and (2.11) holds.

Now let X_1, X_2, X_3 be i.i.d. r.v.'s with distribution function

$$F(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ \sqrt{x} & \text{if} \quad 0 < x \leq 1 \\ 1 & \text{if} \quad x > 1 \end{cases}.$$

Probability density function is equal to

$$f(x) = \begin{cases} 0 & \text{if} \quad x \notin (0,1] \\ \dfrac{1}{2\sqrt{x}} & \text{if} \quad x \in (0,1] \end{cases}.$$

Consider the i.i.d. random variables $X_1' = X_1+c$, $X_2' = X_2+c$, $X_3' = X_3+c$, where c is the solution of the equation

$$\pi + 2\pi c\sqrt{c} - 3\pi c = 3.$$

It is clear that $0<c<1$. Let F_1 be the distribution function of X_1'. Then $F_1(x) = F(x-c)$. It is clear that $F_1 \notin \Im_1$. But one can show that

$$P\{X_3' < X_1' + X_2'\} = 1-\frac{1}{2^2}.$$

In fact, consider

$$\begin{aligned} P\{X_3' < X_1'+X_2'\} &= P\{X_3+c < X_1+c+X_2+c\} = 1-P\{X_1+X_2 < X_3-c\} \\ &= 1-\int_0^1 F^*(x-c)dF(x) = 1-\int_{-c}^{1-c} \frac{1}{2\sqrt{t+c}} F^*(t)dt, \end{aligned}$$

where $F^*(t) = P\{X_1 + X_2 \le t\}$. Let $f^*(t) = \frac{d}{dt}F^*(t)$. It is true that

$$f^*(x) = \int f(x-t)f(t)dt = \int_0^1 f(x-t)\frac{1}{2\sqrt{t}}dt \underset{x-t=z}{=} \int_{x-1}^x \frac{1}{2\sqrt{x-z}}f(z)dz$$
$$= \begin{cases} 0, & x \notin (0,2] \\ \displaystyle\int_0^x \frac{1}{2\sqrt{x-z}}\frac{1}{2\sqrt{z}}dz, & 0 < x < 1 \\ \displaystyle\int_{x-1}^1 \frac{1}{2\sqrt{x-z}}\frac{1}{2\sqrt{z}}dz, & 1 \le x \le 2 \end{cases}$$
$$= \begin{cases} 0, & x \notin (0,2] \\ \dfrac{\pi}{4}, & 0 < x < 1 \\ \dfrac{1}{2}\arcsin\left(\dfrac{2}{x} - 1\right), & 1 \le x \le 2 \end{cases}$$

and

$$F^*(t) = \begin{cases} 0, & x \le 0 \\ \dfrac{\pi}{4}x, & 0 < x < 1 \\ \dfrac{1}{2}\displaystyle\int_0^x \arcsin\left(\frac{2}{t} - 1\right)dt, & 1 \le x \le 2 \\ 1, & x \ge 2 \end{cases}.$$

It is clear that

$$\int_{-c}^{1-c} \frac{1}{2\sqrt{t+c}}F^*(t)dt = \frac{\pi}{4}\int_0^{1-c} \frac{t}{\sqrt{t+c}}dt \underset{\sqrt{t+c}=z}{=} \frac{2\pi}{8}\int_{\sqrt{c}}^1 \frac{(z^2-c)zdz}{z}$$
$$= \frac{\pi}{12} + \frac{\pi}{6}c\sqrt{c} - \frac{\pi}{4}c.$$

Therefore we have

$$P\{X_3' < X_1' + X_2'\} = 1 - \left(\frac{\pi}{12} + \frac{\pi}{6}c\sqrt{c} - \frac{\pi}{4}c\right) = 1 - \frac{\pi + 2\pi c\sqrt{c} - 3\pi c}{12}$$
$$= 1 - \frac{3}{12} = 1 - \frac{1}{2^2}.$$

Remark 2.2. Let X be a nonnegative continuous random variable with d.f. F. We say that F has an increasing failure rate average (IFRA) if for all $x \ge 0$ and $0 < \alpha \le 1$

$$\bar{F}(x) \le (\bar{F}(\alpha x))^{1/\alpha}. \tag{2.12}$$

Note that the exponential distribution is the only life distribution that attains equality in (2.12) for all $x \ge 0$.

Let $\bar{F}(\alpha(x+y)) \ge (\bar{F}(x))^\alpha(\bar{F}(y))^\alpha$, $0 < \alpha \le \frac{1}{2}$. Taking $x = y = \frac{t}{2}$ we have $\bar{F}(\alpha t) \ge (\bar{F}(\frac{t}{2}))^\alpha(\bar{F}(\frac{t}{2}))^\alpha = (\bar{F}(\frac{t}{2}))^{2\alpha}$ which implies $\bar{F}(\beta z) \ge (\bar{F}(z))^\beta$, $0 < \beta \le 1$, $z \ge 0$. Therefore F is IFRA.

3. Characterizations by record values

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed random variables with continuous distribution function F. Define a sequence of record times $U(n)$, $n \geq 1$ as follows: $U(1) = 1$, $U(n) = \min\{j : j > U(n-1), X_j > X_{U(n-1)}\}$, $n > 1$. Let $X_{U(n)}$ be upper record values, $n = 1, 2, \ldots$. For more details on the theory of order statistics and records one can see Galambos (1987), Resnik (1987), Nagaraja (1988), Nevzorov (1988), David (1991), Ahsanullah (1995), among others.

Let $X_{U(n)+1}, X_{U(n)+2}, \ldots$ be the next observations that come after $X_{U(n)}$. It is not difficult to prove that $X_{U(n)}, X_{U(n)+1}, X_{U(n)+2}, \ldots$, are mutually independent and $X_{U(n)+k}$ has the same distribution F for any $k = 1, 2, \ldots$. Let us define the following r.v.'s for a given n:

$$\eta_n(i) = \begin{cases} 1 & \text{if} \quad X_{U(n)+i} < X_{U(n)} \\ 0 & \text{if} \quad X_{U(n)+i} \geq X_{U(n)} \end{cases}, \qquad i = 1, 2, \ldots.$$

As a consequence of Theorem 2.1 we have the following.

THEOREM 3.1. *Let X be a nonnegative r.v. having continuous d.f. F satisfying* $\inf\{x : F(x) > 0\} = 0$. *Then the following statements are equivalent*

(*a*) *X has an exponential distribution with density as given in* (2.3)

(*b*) *for some $n > 1$*

$$E\xi_n(1) = E\eta_n(1)$$

and F is either NBU or NWU.

PROOF. It is not difficult to see that if F is continuous d.f. then (see Bairamov (1997))

$$P\{X_{U(n)+1} < X_{U(n)}\} = 1 - \frac{1}{2^n}.$$

By assumption of the theorem, $F \in \Im_1$ and

$$P\{X_{n+1} < X_1 + X_2 + \cdots + X_n\} = 1 - \frac{1}{2^n}$$

and from Theorem 2.1, F is exponential.

Tata (1969) proved that if $Y_1, Y_2, \ldots, Y_n, \ldots$ are independent and identically distributed random variables with the d.f. $F_0(x) = 1 - \exp(-x)$, $x > 0$ then the r.v.'s $Z_1 = Y_1$, $Z_2 = Y_{U(2)} - Y_1, \ldots, Z_n = Y_{U(n)} - Y_{U(n-1)}$ are independent and

$$P\{Z_n \leq x\} = F_0(x), \qquad (n = 1, 2, \ldots).$$

It is also true that for i.i.d. random variables if the differences $X_{U(n)} - X_{U(n-1)}$, $n \geq 2$ are independent, then the population is exponential (see also Galambos (1987) p. 361).

This theorem implies the following representation

$$Y_{U(n)} \stackrel{d}{=} Y_1 + Y_2 + \cdots + Y_n, \qquad n = 1, 2, \ldots$$

where $\stackrel{d}{=}$ denotes equality in distribution.

THEOREM 3.2. *Let $X_1, X_2, \ldots, X_n, \ldots$ be the sequence of i.i.d. r.v.'s with absolutely continuous d.f. F satisfying* $\inf\{x : F(x) > 0\} = 0$. *Then the following properties are equivalent*

(*a*) *X_1 has an exponential distribution with density as given in* (2.3)

(*b*) *For some $n > 1$*

$$X_1 + X_2 + \cdots + X_n \stackrel{d}{=} X_{U(n)} \quad \text{and} \quad F \in \Im_1.$$

PROOF. It is clear that $(a) \Rightarrow (b)$. Now we will prove other implication.

Let $X_1 + X_2 + \cdots + X_n \stackrel{d}{=} X_{U(n)}$ and $F \in \Im_1$. Consider

$$\begin{aligned} P\{F_0(Y_1 + Y_2 + \cdots + Y_n) \le t\} &= P\{Y_1 + Y_2 + \cdots + Y_n \le F_0^{-1}(t)\} \\ &= \frac{1}{(n-1)!} \int_0^{-\ln(1-t)} e^{-x} x^{n-1} dx \\ &= \frac{1}{(n-1)!} \int_0^t \left[\ln \frac{1}{1-x}\right]^{n-1} dx. \end{aligned} \tag{3.1}$$

We have

$$\begin{aligned} P\{X_1 + X_2 + \cdots + X_n \le u\} &= P\{X_{U(n)} \le u\} \\ &= \frac{1}{(n-1)!} \int_0^{-\ln(1-F(u))} e^{-x} x^{n-1} dx. \end{aligned} \tag{3.2}$$

Taking $F(u) = t$, $u = F^{-1}(t)$ in (3.2) we obtain

$$P\{X_1 + X_2 + \cdots + X_n \le F^{-1}(t)\} = \frac{1}{(n-1)!} \int_0^{-\ln(1-t)} e^{-x} x^{n-1} dx. \tag{3.3}$$

From (3.1) and (3.3) we have

$$P\{F(X_1 + X_2 + \cdots + X_n) \le t\} = P\{F_0(Y_1 + Y_2 + \cdots + Y_n) \le t\}, \quad t \in [0,1]. \tag{3.4}$$

Then one can show that

$$P\{X_{n+1} < X_1 + X_2 + \cdots + X_n\} = P\{Y_{n+1} < Y_1 + Y_2 + \cdots + Y_n\} = 1 - \frac{1}{2^n}. \tag{3.5}$$

In fact, one has

$$\begin{aligned} &P\{X_{n+1} < X_1 + X_2 + \cdots + X_n\} \\ &\quad = \int \cdots \int F(u_1 + u_2 + \cdots + u_n) dF(u_1) \cdots dF(u_n) \\ &\quad = EF(X_1 + X_2 + \cdots + X_n) = \int x dP\{F(X_1 + X_2 + \cdots + X_n) \le x\}, \\ &P\{Y_{n+1} < Y_1 + Y_2 + \cdots + Y_n\} = \int x dP\{F_0(Y_1 + Y_2 + \cdots + Y_n) \le x\}. \end{aligned}$$

Then by using (3.4) one can obtain (3.5). From Theorem 2.1 F is exponential, which concludes the proof.

4. The relation equal in s-sense and its application to the characterization problem

Denote by $\boldsymbol{F}_c$ the class of all continuous d.f's. Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v.'s with d.f. F, $F \in \boldsymbol{F}_c$. Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be the order statistics constructed by the sample $X_1, X_2, \ldots, X_n$. Due to a result referred to as the probability integral transformation the random variable $F(X)$ has uniform distribution on (0,1) and the random variable $F(X_{i:n})$, $1 \le i \le n$ has a beta distribution with parameters $\alpha = i$, $\beta = n - i + 1$ (see Randles and Wolfe (1979)). That is, the d.f.'s of random variables $F(X)$ and $F(X_{i:n})$ are not depend on F.

It is also known that if $X_1, X_2, \ldots, X_n$ is the sample from continuous distribution with d.f. F and if $f(x_1, x_2, \ldots, x_n)$ is some continuous and symmetric function of n arguments, then the r.v. $F(f(X_1, X_2, \ldots, X_n))$ has the same d.f. for all $F \in \boldsymbol{F}_c$, only if $f(X_1, X_2, \ldots, X_n) = X_{i:n}$, $1 \le i \le n$ (see Robbins (1944)).

Let $F \in \boldsymbol{F}_c$. Consider $\Im = \{F_\theta(x) : F_\theta(x) = F(\frac{x}{\theta}), \theta \in \Theta\}$, $\Im \subset \boldsymbol{F}_c$ and $f(X_1, X_2, \ldots, X_n) = \sum_{i=1}^n a_i X_{i:n}$, ($a_i (1 \le i \le n)$ are constants). It is clear that if $X_1, X_2, \ldots, X_n$ has d.f. $F_\theta \in \Im$, then

$$F_\theta\left(\sum_{i=1}^n a_i X_{i:n}\right) = F\left(\sum_{i=1}^n a_i \frac{X_{i:n}}{\theta}\right) \stackrel{d}{=} F\left(\sum_{i=1}^n a_i Z_{i:n}\right),$$

where $Z_{1:n} \le Z_{2:n} \le \cdots \le Z_{n:n}$ are order statistics constructed by the sample $Z_1, Z_2, \ldots, Z_n$ with d.f. F. Hence the d.f. of r.v. $F_\theta(f(X_1, X_2, \ldots, X_n)) = F_\theta(\sum_{i=1}^n a_i X_{i:n})$ is independent of θ.

As it is shown here for the class $\Im \subset \boldsymbol{F}_c$, there exists a function $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^n a_i x_{i:n}$ (which is not equal identically to the function $\varphi_i(x_1, x_2, \ldots, x_n) = x_{(i)}$, where $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$, $(x_1, x_2, \ldots, x_n) \in R^n$), such that d.f. of r.v. $F(f(X_1, X_2, \ldots, X_n))$ (when $X_1, X_2, \ldots, X_n$ have d.f. F) is the same for all $F \in \Im$. It is also possible other examples.

DEFINITION 4.1. Let X and Z be random variables with continuous d.f. F and G, respectively. Let $n \ge 1$ be some integer, $X_1, X_2, \ldots, X_n$ and $Z_1, Z_2, \ldots, Z_n$ be independent copies of X and Z, respectively. We say "X and Z are equal in s_n-sense" (or F and G are equal in s_n-sense) and denote $X \stackrel{s_n}{=} Z$ (or $F \stackrel{s_n}{=} G$) if

$$P\{F(X_1 + X_2 + \cdots + X_n) \le x\} = P\{G(Z_1 + Z_2 + \cdots + Z_n) \le x\}, \qquad \text{for all} \quad x \in R.$$

It is clear that $\stackrel{s_n}{=}$ is an the equivalence relation.

Let us denote by $\Im_F^*$ the class of all continuous d.f.'s which are equal to F in s_n-sense. It is clear that if $G \in \Im_F^*$, then $F \in \Im_G^*$. So $F \stackrel{s_n}{=} G$ iff $\Im_F^* = \Im_G^*$.

Now let $X_1, X_2, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ and $X_1', X_2', \ldots, X_n', X_{n+1}', X_{n+2}', \ldots, X_{n+m}'$ be i.i.d. r.v.'s with d.f.'s F and G respectively. Denote

$$\xi_k(n) = \begin{cases} 1 & \text{if} \quad X_{n+k} < \sum_{i=1}^n X_i \\ 0 & \text{otherwise} \end{cases}, \qquad \eta_k(n) = \begin{cases} 1 & \text{if} \quad X_{n+k}' < \sum_{i=1}^n X_i' \\ 0 & \text{otherwise} \end{cases},$$

$$k = 1, 2, \ldots, m$$

and

$$S_m(n) = \prod_{k=1}^m \xi_k(n), \qquad T_m(n) = \prod_{k=1}^m \eta_k(n).$$

THEOREM 4.1. *Let F and G be absolutely continuous with respect to Lebesque measure and strictly increasing d.f.'s. Then $F \in \Im_G^*$ if and only if*

$$ES_m(n) = ET_m(n), \qquad m = N_1, N_2, \ldots \sum_i N_i^{-1} = \infty.$$

PROOF. Let $F \overset{s_n}{=} G$. Then

$$\begin{aligned} ES_m(n) &= P\left\{X_{n+1} < \sum_{i=1}^n X_i, X_{n+2} < \sum_{i=1}^n X_i, \ldots, X_{n+m} < \sum_{i=1}^n X_i\right\} \\ &= \int \cdots \int \left(F\left(\sum_{i=1}^n x_i\right)\right)^m dF(x_1) \cdots dF(x_n) \\ &= E\left[\left(F\left(\sum_{i=1}^n X_i\right)\right)^m\right] = \int x^m dP\left\{F\left(\sum_{i=1}^n X_i\right) \le x\right\} \end{aligned}$$

and analogously

$$ET_m(n) = \int x^m dP\left\{G\left(\sum_{i=1}^n X_i'\right) \le x\right\},$$

therefore $ES_m(n) = ET_m(n)$, $m = 0, 1, 2, \ldots$.

Now let $ES_m(n) = ET_m(n)$, $m = N_1, N_2, \ldots$, $\sum_i N_i^{-1} = \infty$. One can write

$$E\left[\left(F\left(\sum_{i=1}^n X_i\right)\right)^m\right] = E\left[\left(G\left(\sum_{i=1}^n X_i'\right)\right)^m\right]$$

and

$$\int_0^1 x^m dP\left\{F\left(\sum_{i=1}^n X_i\right) \le x\right\} = \int_0^1 x^m dP\left\{G\left(\sum_{i=1}^n X_i'\right) \le x\right\},$$

$$m = N_1, N_2, \ldots$$

According to Müntz -Szasz theorem (see Akhieser (1961)) the sequence of functions $\{x^m\}$, $m = N_1, N_2, \ldots, \sum_i N_i^{-1} = \infty$ is complete in the space of all continuous on $[0,1]$ functions $C_{[0,1]}$. Therefore

$$P\left\{F\left(\sum_{i=1}^n X_i\right) \le x\right\} = P\left\{G\left(\sum_{i=1}^n X_i'\right) \le x\right\}, \qquad \text{for all} \quad x \text{ and } F \overset{s_n}{=} G.$$

Remark 4.1. Consider the exponential distribution function $F_\theta(x) = 1 - e^{-x/\theta}$, $x \ge 0$, $\theta > 0$. From (2.4) we have $F_o \overset{s_n}{=} F_\theta$ for all $\theta > 0$, so $\Im_{F_0}^* = \Im_{F_\theta}^*$ for all $\theta > 0$, where $F_o(x)$ denotes the exponential d.f. with $\theta = 1$.

LEMMA 4.1. *Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. r.v.'s with d.f. F. Then*

$$X_1 + X_2 + \cdots + X_n \overset{d}{=} X_{U(n)}$$

if and only if

$$F \overset{s_n}{=} F_0,$$

where $F_0(x) = 1 - e^{-x}$, $x \geq 0$.

PROOF. One can write

$$\begin{aligned}
&X_1 + X_2 + \cdots + X_n \stackrel{d}{=} X_{U(n)} \Leftrightarrow \\
&P\{X_1 + X_2 + \cdots + X_n \leq u\} = \frac{1}{(n-1)!}\int_0^{-\ln(1-F(u))} e^{-x}x^{n-1}dx \Leftrightarrow \\
&P\{X_1 + X_2 + \cdots + X_n \leq F^{-1}(u)\} = \frac{1}{(n-1)!}\int_0^{-\ln(1-u)} e^{-x}x^{n-1}dx \\
&\qquad\qquad = P\{Y_1 + \cdots + Y_n \leq -\ln(1-u)\} \Leftrightarrow \\
&P\{X_1 + X_2 + \cdots + X_n \leq F^{-1}(u)\} = P\{Y_1 + Y_2 + \cdots + Y_n \leq F_0^{-1}(u)\}.
\end{aligned}$$

THEOREM 4.2. *Let* $X_1, X_2, \ldots, X_n, \ldots$ *be a sequence of i.i.d. r.v.'s with absolutely continuous d.f.'s* F. *Then*

$$X_1 + X_2 + \cdots + X_n \stackrel{d}{=} X_{U(n)}$$

if and only if

$$ES_m(n) = \sum_{i=0}^{m}(-1)^i\frac{\binom{m}{i}}{(i+1)^n}, \qquad m = N_1, N_2, \ldots, \qquad \sum_i N_i^{-1} = \infty.$$

PROOF. Since $Y_1 + Y_2 + \cdots + Y_n$ has a gamma distribution with p.d.f. $g_{1,n}(x)$ one obtain

$$\begin{aligned}
&P\left\{Y_{n+1} < \sum_{i=1}^{n} Y_i, Y_{n+2} < \sum_{i=1}^{n} Y_i, \ldots, Y_{n+m} < \sum_{i=1}^{n} Y_i\right\} \\
&\quad = \frac{1}{(n-1)!}\int_0^{\infty} y^{n-1}e^{-y}(1-e^{-y})^m dy \\
&\quad = \frac{1}{(n-1)!}\int_0^1 (1-x)^m\left[\ln\frac{1}{x}\right]^{n-1}dx = \frac{1}{(n-1)!}\int_0^1 \sum_{i=0}^{m}(-1)^i\binom{m}{i}x^i\left[\ln\frac{1}{x}\right]^{n-1}dx \\
&\quad = \frac{1}{(n-1)!}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\frac{\Gamma(n)}{(i+1)^n} = \sum_{i=0}^{m}(-1)^i\frac{\binom{m}{i}}{(i+1)^n}
\end{aligned}$$

(see Dwight (1961) 863.04). From this fact and Theorem 4.1, $ES_m(n) = \sum_{i=0}^{m}(-1)^i \frac{\binom{m}{i}}{(i+1)^n}$, $m = N_1, N_2, \ldots$, $\sum_i N_i^{-1} = \infty$ is equivalent to $F \in \Im_{F_0}^*$. The proof is completed using Lemma 4.1.

THEOREM 4.3. *Let* X_n, $n \geq 1$ *be a sequence of independent random variables with common absolutely continuous d.f.* $F(x)$ $(F'(x) = f(x))$. *Let the variables* X_j *be positive with probability one. Assume also that the density function* $f(x)$ *is strictly positive for* $x > 0$ *and the limit*

$$\lim_{x\to+0}\frac{F(x)}{x}$$

exists and is finite. Under these assumptions $F(x)$ is exponential d.f. iff

$$ES_m(2) = \sum_{i=0}^{m}(-1)^i \frac{\binom{m}{i}}{(i+1)^2} = \frac{1}{m+1}\sum_{i=1}^{m+1}\frac{1}{i}, \qquad m = N_1, N_2, \ldots, \qquad \sum_i N_i^{-1} = \infty.$$

PROOF. Under assumptions of theorem it is known that X has an exponential distribution iff $X_1 + X_2 \stackrel{d}{=} X_{U(2)}$ (see Kakosyan *et al.* (1984) p. 126). Theorem 4.2 concludes the proof.

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